Name: Solutions

Math 106 Section 550 Exam 1

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it! You should solve each problem using the methods explored up to this point in class and the text, together with any necessary algebra, trigonometry, or geometry, as appropriate.

1. (10 pts.) Determine the following limit:

$$\lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{2x+1} - \frac{1}{x+4}\right)$$

$$= \lim_{x\to 3} \frac{1}{x-3} \frac{(x+4)-(2x+1)}{(2x+1)(x+4)} = \lim_{x\to 3} \frac{x+4-2x-1}{(x-3)(2x+1)(x+4)}$$

$$= \lim_{x\to 3} \frac{-x+3}{(x-3)(2x+1)(x+4)} = \lim_{x\to 3} \frac{-(x-3)}{(x-3)(2x+1)(x+4)} = \lim_{x\to 3} \frac{-1}{(2x+1)(x+4)}$$

$$= \frac{-1}{(2\cdot 3+1)(3+4)} = \frac{-1}{7\cdot 7} = \frac{-1}{49}$$

2. (10 pts. each) Find, using any method, the derivatives of the following functions:

(a): $f(x) = \dfrac{x^4+3x-4}{2x^2-3x+1}$

$$f'(x) = \frac{(x^4+3x-4)'(2x^2-3x+1) - (x^4+3x-4)(2x^2-3x+1)'}{(2x^2-3x+1)^2}$$

$$= \frac{(4x^3+3)(2x^2-3x+1) - (x^4+3x-4)(4x-3)}{(2x^2-3x+1)^2}$$

(b): $g(x) = x^3 \sin x - 3x^{-4}$

$$g'(x) = (x^3)'(\sin x) + (x^3)(\sin x)' - 3(x^{-4})'$$

$$= 3x^2 \sin x + x^3 \cos x - 3(-4x^{-5})$$

3. (15 pts.) If u and v are functions of x, $u(1) = 3, v(1) = -2, u'(1) = 1$, and $v'(1) = 7$, what is the derivative of $3uv - 2xu$ at $x = 1$?

$$(3uv-2xu)' = 3(uv)' - 2(xu)'$$
$$= 3(u'v+uv') - 2(1\cdot u + xu')$$

So $(3uv-2xu)'(1) = 3(u'(1)v(1)+u(1)v'(1)) - 2(u(1)+1\cdot u'(1))$

$$= 3(1\cdot(-2)+3\cdot 7) - 2(3+1)$$
$$= 3(-2+21) - 2(4) = 3(19) - 8$$
$$= 57 - 8 = \cancel{8}\,49$$

4. (15 pts.) Let $f(x) = x^3 + 3x^2 - 2x - 7$.

Find an interval $[a, b]$ of length one that you can guarantee includes a root of $f(x)$ (i.e., a c in $[a, b]$ with $f(c) = 0$). [Hint: Try to find one somewhere between, oh, I don't know, -3 and 3?]

$f(-3) = (-3)^3 + 3(-3)^2 - 2(-3) - 7 = -27+27+6-7 = -1 < 0$

$f(-2) = (-2)^3 + 3(-2)^2 - 2(-2) - 7 = -8+12+4-7 = 16-15 = 1 > 0$

$f(-1) = (-1)^3 + 3(-1)^2 - 2(-1) - 7 = -1+3+2-7 = 5-8 = -3 < 0$

$f(0) = 0^3 + 3(0)^2 - 2(0) - 7 = -7 < 0$

$f(1) = 1^3 + 3(1)^2 - 2(1) - 7 = 1+3-2-7 = 4-9 = -5 < 0$

$f(2) = 2^3 + 3(2)^2 - 2(2) - 7 = 8+12-4-7 = 20-11 = 9 > 0$

$f(3) = 3^3 + 3(3)^2 - 2(3) - 7 = 27+27-6-7 = 54-13 = 41 > 0$

Since f is continuous (it is a polynomial) and 0 is between $f(-3)$ and $f(-2)$, the Intermediate Value Theorem guarantees that there is a root of f between -3 and -2, ie, in $[-3,-2]$

In fact, since 0 is between $f(-2)$ and $f(-1)$, and between $f(1)$ and $f(2)$, it also guarantees that there is a root ~~be~~ in $[-2,-1]$, and in $[1,2]$

5. (15 pts.) (a): Find, using (one of) the (limit) definitions of the derivative, the derivative of the function

$$f(x)=3x^2-5x+11 \qquad f(1)=3-5+11=14-5=9$$

at the point x=1.

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{(3x^2-5x+11)-9}{x-1}$$

$$=\lim_{x\to 1}\frac{3x^2-5x+2}{x-1}=\lim_{x\to 1}\frac{\cancel{(x-1)}(3x-2)}{\cancel{x-1}}$$

$$=\lim_{x\to 1}3x-2=3\cdot 1-2=3-2=1.$$

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or

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{1}{h}\left(3(1+h)^3-5(1+h)+11)-9\right)$$

$$=\lim_{h\to 0}\frac{1}{h}(3h^3+6h+3-5h-5+11-9)$$

$$=\lim_{h\to 0}\frac{1}{h}(3h^3+h)=\lim_{h\to 0}3h^2+1=3(0)^2+1=1$$

[or: compute $f'(x)$ (from the definition), and plug in $x=1$.]

(b): (5 pts.) Find the equation for the tangent line to the graph of y=f(x)=$3x^2-5x+11$ at the point (1,f(1)).

slope $=f'(1)=1$, point $=(1,f(1))=(1,9)$

$$y-9=1(x-1)=x-1$$

$$y=x-1+9=x+8$$

$$\boxed{y=x+8}$$

(or: $y-9=1(x-1)$ is fine...)

6. (20 pts.) Find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{2x^2 + 3x - 15}{(x+2)^2}$$

and determine the limiting values of f as x approaches each side of your vertical asymptotes.

Vertical: check denom $= 0$ $\quad (x+2)^2 = 0$

$x+2=0$, $x=-2$

numerator at -2: $2(-2)^2+3(-2)-15 = 8-6-15 = -13 < 0$

so $\boxed{x=-2 \text{ is a vertical asymptote}}$.

As $x \to -2^-$ $f(x)$ is $\dfrac{\approx -13}{(\text{small pos})}$

so $f(x) = $ large neg.,

$\boxed{\lim_{x\to -2^-} f(x) = -\infty}$.

numer $\quad -13$

-2

denom $\quad + + + \; 0 \; + + +$

As $x \to -2^+$, $f(x)$ is $\dfrac{\approx -13}{(\text{small pos})}$, so

$f(x) = $ large neg., $\boxed{\lim_{x\to -2^+} f(x) = -\infty}$

horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2+3x-15}{(x+2)^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}(2x^2+3x-15)}{\frac{1}{x^2}(x+2)^2}$$

$$= \lim_{x\to\infty} \frac{2+3(\frac{1}{x})-15(\frac{1}{x})^2}{(1+2(\frac{1}{x}))^2} = \frac{2+3(0)-15(0)^2}{(1+2(0))^2} = \frac{2}{1} = 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2+3(\frac{1}{x})-15(\frac{1}{x})^2}{(1+2(\frac{1}{x}))^2} = \frac{2+3(0)-15(0)^2}{(1+2(0))^2} = \frac{2}{1} = 2$$

So $\boxed{y=2 \text{ is the only horizontal asymptote}}$.