Practice problem solutions

#1. $\lim_{x\to 2} \frac{x^3-x^2-8}{2x^2+3x-14} = (\text{plug in})\left[\frac{8-4-8}{8+6-14} = \frac{-4}{0} \text{ undefined}\right]$ $\underline{\underline{\text{DNE}}}$.

(Should have been:

$$\lim_{x\to 2} \frac{x^3-x^2-4}{2x^2+3x-14} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+x-2)}{\cancel{(x-2)}(2x+7)}$$

$$= \lim_{x\to 2} \frac{x^2+x-2}{2x+7} = \frac{4+2-2}{4+7} = \frac{4}{11}$$

$$\lim_{x\to 0} \frac{x\sin x - \cos x}{x \sin x + \cos x} = (\text{plug in}) \frac{1\cdot 0 - 1}{1\cdot 0 + 1} = \frac{-1}{1} = -1$$

This is the limit b/c x, $\sin x$, and $\cos x$ are continuous at 0, so sums, diffs, prods, quots are also cts; eval limit as value at 0.

#2: $f(x) = \frac{2x+15}{(x-3)^3}$ denom $=0$ at $x=3$ numerator $= 6+15 = 21$ at $x=3$

so $x=3$ is a vertical asymptote.

$\lim_{x\to 3^-} \frac{2x+15}{(x-3)^3}$: $\frac{21}{(\text{small neg})^3} = \frac{21}{\text{small neg}}$ = big neg; limit $= -\infty$

$\lim_{x\to 3^+} \frac{2x+15}{(x-3)^3}$: $\frac{21}{(\text{small pos})^3} = \frac{21}{\text{small pos}}$ = big pos.; limit $= +\infty$

$$\lim_{x\to\infty} \frac{2x+15}{(x-3)^3} = \lim_{x\to\infty} \frac{\frac{1}{x^3}(2x+15)}{\frac{1}{x^3}(x-3)^3} = \lim_{x\to\infty} \frac{2(\frac{1}{x})^2+15(\frac{1}{x})^3}{(1-3(\frac{1}{x}))^3} = \frac{0+0}{1^3} = 0$$

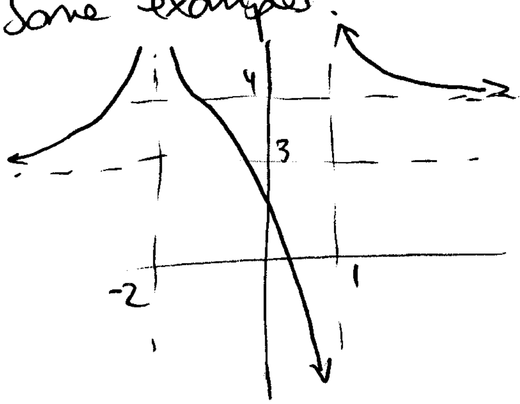
$$\lim_{x\to -\infty} \frac{2x+15}{(x-3)^3} = \lim_{x\to -\infty} \frac{2(\frac{1}{x})^2+15(\frac{1}{x})^3}{(1-3(\frac{1}{x}))^3} = \frac{0+0}{1^3} = 0$$

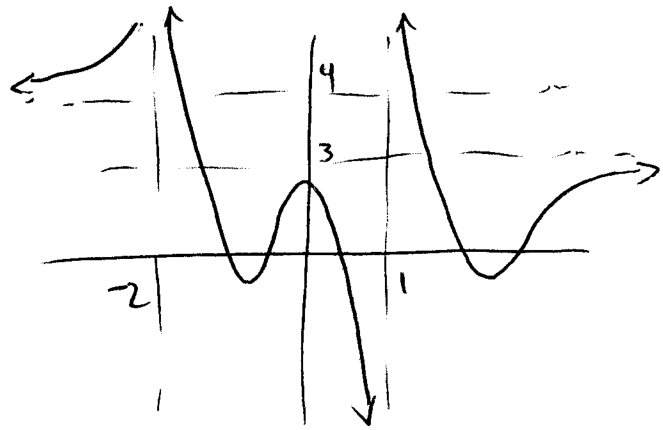
$y=0$ is the only horizontal asymptote.

#3: $y=3$, $y=4$ horiz asymptotes

$f(x) \to \infty$ as $x \to -2^-, -2^+, 1^+$, $f(x) \to -\infty$ as $x \to 1^-$

Some examples:





#4 $f(x) = x^4 + 2x^3 + 7x + 5$ has 2 roots in $[-3,3]$

$f(-3) = 81 - 2\cdot 27 - 21 + 5 = 81 - 54 - 21 + 5 = 11 > 0$

$f(-2) = 16 - 8 - 14 + 5 = 21 - 22 = -1 < 0$

$f(-1) = 1 - 2 - 7 + 5 = 6 - 9 = -3 < 0$

$f(0) = 0 + 0 + 0 + 5 = 5 > 0$

Since f is continuous and 0 is b/w $f(-3)$ and $f(-2)$, there is a root ($f(c)=0$) in $[-3,-2]$

Since 0 is b/w $f(-1)$ and $f(0)$, there is also a root in $[-1,0]$. So f has at least 2 roots in $[-3,3]$.

#5

$$f(x) = \begin{cases} x & x \le -2 \\ x^2-6 & -2 \le x < 1 \\ 3 & x = 1 \\ 2x-7 & 1 < x < 4 \\ x^2 & 4 \le x \end{cases}$$

The pieces are cts, so the only possible discontinuities are at $x=-2$, $x=1$, or $x=4$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} x = -2 = f(-2) = \lim_{x\to -2^+} x^2-6 = \lim_{x\to -2^+} f(x)$$

so f is cts at -2

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2-6 = -5 \neq 3 = f(1) \neq -5 = \lim_{x\to 1^+} 2x-7 = \lim_{x\to 1^+} f(x)$$

Since $\lim\limits_{x\to 1^-} f(x) = -5 = \lim\limits_{x\to 1^+} f(x)$, f has a removable discontinuity at $x=1$.

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 2x-7 = 1 \neq 16 = f(4) = \lim_{x\to 4^+} x^2 = \lim_{x\to 4^+} f(x).$$

Since $\lim\limits_{x\to 4^-} f(x) \neq \lim\limits_{x\to 4^+} f(x)$, but both exist, f has a jump discontinuity at $x=4$.

#6

$f(x) = x^3-3x+10$

$f(2) = 8-6+10 = 12$

$$f'(2) = \lim_{x\to 2} \frac{(x^3-3x+10)-12}{x-2} = \lim_{x\to 2} \frac{x^3-3x-2}{x-2}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+1)}{\cancel{x-2}} = \lim_{x\to 2} x^2+2x+1 = 9.$$

tangent line: slope $=9$, point $=(2,12)$

$$y-12 = 9(x-2) = 9x-18 \quad \boxed{y = 9x-6}$$

$f(x) = \frac{2}{x-3}$ at $x=4$ $\quad f(4) = \frac{2}{4-3} = 2$

$$f'(4) = \lim_{x\to 4} \frac{\frac{2}{x-3} - 2}{x-4} = \lim_{x\to 4} \frac{2-(2(x-3))}{(x-3)(x-4)}$$

$$= \lim_{x\to 4} \frac{8-2x}{(x-3)(x-4)} = \lim_{x\to 4} \frac{-2\cancel{(x-4)}}{(x-3)\cancel{(x-4)}} = \lim_{x\to 4} \frac{-2}{x-3} = \frac{-2}{1} = -2$$

tangent line: slope $= -2$, point $= (4,2)$

$$y-2 = -2(x-4) = -2x+8 \quad \boxed{y=-2x+10}$$

#7

$$f(x) = 5x^{-3/2} + \frac{1}{x^2-x+3}$$

$$f'(x) = 5\left(-\tfrac{3}{2}x^{-5/2}\right) + \frac{(0)(x^2-x+3)-(1)(2x-1)}{(x^2-x+3)^2}$$

$$g(x) = x^{1/2}(x^2+3x-3) = x^{5/2} + 3x^{3/2} - 3x^{1/2}$$

$$g'(x) = \tfrac{5}{2}x^{3/2} + 3\left(\tfrac{3}{2}x^{\frac{1}{2}}\right) - 3\left(\tfrac{1}{2}x^{-1/2}\right)$$

or $\quad g'(x) = \left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)(x^2+3x-3) + \left(x^{\frac{1}{2}}\right)(2x+3)$.

$$h(x) = \sin^2 x \cos x$$

$$h'(x) = (\sin x \sin x)' \cos x + (\sin^2 x)(-\sin x)$$

$$= ((\cos x)(\sin x) + (\sin x)(\cos x))\cos x - \sin^3 x$$

$$= 2\sin x \cos^2 x - \sin^3 x$$

#8. $A(2)=1$, $B(2)=-2$ $\quad A'(2)=3$ $\quad B'(2)=4$

$$\left(\frac{A}{5+B}\right)' = \frac{(A')(5+B)-(A)(B')}{(5+B)^2},$$

so $$\left(\frac{A}{5+B}\right)'(2) = \frac{A'(2)(5+B(2)) - A(2)\,B'(2)}{(B(2))^2} = \frac{3(5-2)-1(4)}{(5-2)^2}$$

$$= \frac{9-4}{3^2} = \frac{5}{9}$$

$$(3xAB)' = 3\left(1(AB) + x(AB)'\right)$$
$$= 3\left(AB + x(A'B + AB')\right),$$

so $$(3xAB)'(2) = 3\left(A(2)B(2) + 2\left(A'(2)B(2) + A(2)B'(2)\right)\right)$$
$$= 3\left(1\cdot(-2) + 2\left(3\cdot(-2) + 1\cdot(4)\right)\right)$$
$$= 3\left(-2 + 2(-6+4)\right) = 3\left(-2+2(-2)\right)$$
$$= 3\left(-2+(-4)\right) = 3(-6) = -18.$$