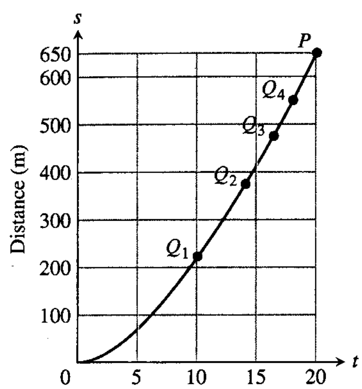
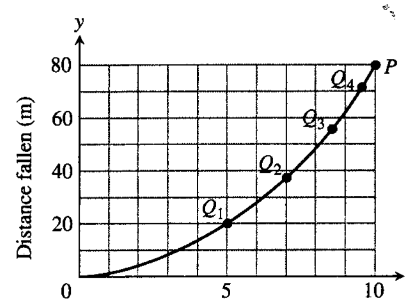
EXERCISES 2.1

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

1. $f(x) = x^3 + 1$;
   a. $[2, 3]$ b. $[-1, 1]$
2. $g(x) = x^2$;
   a. $[-1, 1]$ b. $[-2, 0]$
3. $h(t) = \cot t$;
   a. $[\pi/4, 3\pi/4]$ b. $[\pi/6, \pi/2]$
4. $g(t) = 2 + \cos t$;
   a. $[0, \pi]$ b. $[-\pi, \pi]$
5. $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$
6. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$
7. **Speed of a car** The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



   a. Estimate the slopes of secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?
   b. Then estimate the car's speed at time $t = 20$ sec.
8. The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.
   a. Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Figure 2.6.
   b. About how fast was the object going when it hit the surface?



In Exercises 9–16, use the method in Example 3 to find **(a)** the slope of the curve at the given point P, and **(b)** an equation of the tangent line at P.

9. $y = x^2 - 3$, $P(2, 1)$
10. $y = 5 - x^2$, $P(1, 4)$
11. $y = x^2 - 2x - 3$, $P(2, -3)$
12. $y = x^2 - 4x$, $P(1, -3)$
13. $y = x^3$, $P(2, 8)$
14. $y = 2 - x^3$, $P(1, 1)$
15. $y = x^3 - 12x$, $P(1, -11)$
16. $y = x^3 - 3x^2 + 4$, $P(2, 0)$

T 17. The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in $1000s
2000	6
2001	27
2002	62
2003	111
2004	174

   a. Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.
   b. What is the average rate of increase of the profits between 2002 and 2004?
   c. Use your graph to estimate the rate at which the profits were changing in 2002.

T 18. Make a table of values for the function $F(x) = (x + 2)/(x - 2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.
   a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.
   b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

T **19.** Let $g(x) = \sqrt{x}$ for $x \geq 0$.

a. Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1 + h]$.

b. Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1 + h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001.

c. What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?

d. Calculate the limit as h approaches zero of the average rate of change of $g(x)$ with respect to x over the interval $[1, 1 + h]$.

T **20.** Let $f(t) = 1/t$ for $t \neq 0$.

a. Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.

b. Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001.

c. What does your table indicate is the rate of change of f with respect to t at $t = 2$?

d. Calculate the limit as T approaches 2 of the average rate of change of f with respect to t over the interval from 2 to T. You will have to do some algebra before you can substitute $T = 2$.

2.2 Limit of a Function and Limit Laws

In Section 2.1 we saw that limits arise when finding the instantaneous rate of change of a function or the tangent to a curve. Here we begin with an informal definition of *limit* and show how we can calculate the values of limits. A precise definition is presented in the next section.

HISTORICAL ESSAY*

Limits

Limits of Function Values

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L." Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to x_0 (on either side of x_0). This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch.* To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years.* The definition is clear enough, however, to enable us to recognize and evaluate limits of specific functions. We will need the precise definition of Section 2.3, however, when we set out to prove theorems about limits.

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

*To learn more about the historical figures and the development of the major elements and topics of calculus, visit **www.aw-bc.com/thomas.**