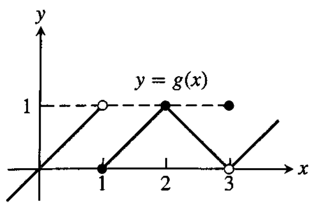
EXERCISES 2.2

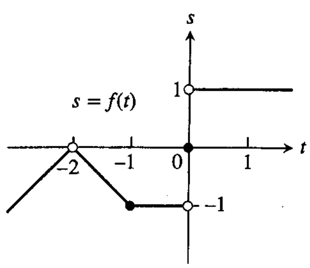
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim\limits_{x\to 1} g(x)$ b. $\lim\limits_{x\to 2} g(x)$ c. $\lim\limits_{x\to 3} g(x)$



2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim\limits_{t\to -2} f(t)$ b. $\lim\limits_{t\to -1} f(t)$ c. $\lim\limits_{t\to 0} f(t)$



3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim\limits_{x\to 0} f(x)$ exists.

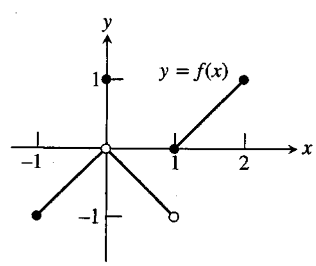
   b. $\lim\limits_{x\to 0} f(x) = 0$.

   c. $\lim\limits_{x\to 0} f(x) = 1$.

   d. $\lim\limits_{x\to 1} f(x) = 1$.

   e. $\lim\limits_{x\to 1} f(x) = 0$.

   f. $\lim\limits_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$.



4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

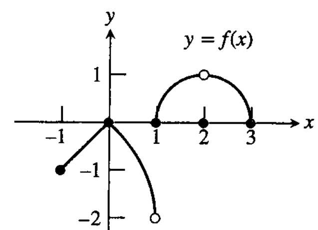
   a. $\lim\limits_{x\to 2} f(x)$ does not exist.

   b. $\lim\limits_{x\to 2} f(x) = 2$.

   c. $\lim\limits_{x\to 1} f(x)$ does not exist.

   d. $\lim\limits_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$.

   e. $\lim\limits_{x\to x_0} f(x)$ exists at every point x_0 in $(1, 3)$.



In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim\limits_{x\to 0} \dfrac{x}{|x|}$

6. $\lim\limits_{x\to 1} \dfrac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = x_0$. Can anything be said about the existence of $\lim_{x\to x_0} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Find the limits in Exercises 11–28.

11. $\lim\limits_{x\to -7} (2x + 5)$

12. $\lim\limits_{x\to 12} (10 - 3x)$

13. $\lim\limits_{x\to 2} (-x^2 + 5x - 2)$

14. $\lim\limits_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim\limits_{t\to 6} 8(t - 5)(t - 7)$

16. $\lim\limits_{s\to 2/3} 3s(2s - 1)$

17. $\lim\limits_{x\to 2} \dfrac{x+3}{x+6}$

18. $\lim\limits_{x\to 5} \dfrac{4}{x-7}$

19. $\lim\limits_{y\to -5} \dfrac{y^2}{5-y}$

20. $\lim\limits_{y\to 2} \dfrac{y+2}{y^2+5y+6}$

21. $\lim\limits_{x\to -1} 3(2x - 1)^2$

22. $\lim\limits_{x\to -4} (x + 3)^{1984}$

23. $\lim\limits_{y\to -3} (5 - y)^{4/3}$

24. $\lim\limits_{z\to 0} (2z - 8)^{1/3}$

25. $\lim\limits_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$

26. $\lim\limits_{h\to 0} \dfrac{5}{\sqrt{5h+4}+2}$

27. $\lim\limits_{h\to 0} \dfrac{\sqrt{3h+1}-1}{h}$

28. $\lim\limits_{h\to 0} \dfrac{\sqrt{5h+4}-2}{h}$

Find the limits in Exercises 29–46.

29. $\lim_{x\to 5} \frac{x-5}{x^2-25}$

30. $\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$

31. $\lim_{x\to -5} \frac{x^2+3x-10}{x+5}$

32. $\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$

33. $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$

34. $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

35. $\lim_{x\to -2} \frac{-2x-4}{x^3+2x^2}$

36. $\lim_{y\to 0} \frac{5y^3+8y^2}{3y^4-16y^2}$

37. $\lim_{u\to 1} \frac{u^4-1}{u^3-1}$

38. $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$

39. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

40. $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}}$

41. $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$

42. $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

43. $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

44. $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$

45. $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3}$

46. $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

Find the limits in Exercises 47–54.

47. $\lim_{x\to 0} (2\sin x - 1)$

48. $\lim_{x\to 0} \sin^2 x$

49. $\lim_{x\to 0} \sec x$

50. $\lim_{x\to 0} \tan x$

51. $\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x}$

52. $\lim_{x\to 0} (x^2-1)(2-\cos x)$

53. $\lim_{x\to 0} \sqrt{x+1}\cos^{1/3} x$

54. $\lim_{x\to 0} \sqrt{1+\cos^2 x}$

55. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to 0}(2f(x)-g(x))}{\lim_{x\to 0}(f(x)+7)^{2/3}} \quad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}\left(f(x)+7\right)\right)^{2/3}} \quad \text{(b)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \quad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

56. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4-r(x)))} \quad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1}(4-r(x))\right)} \quad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x)\right)} \quad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

57. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim_{x\to c} f(x)g(x)$

b. $\lim_{x\to c} 2f(x)g(x)$

c. $\lim_{x\to c} (f(x)+3g(x))$

d. $\lim_{x\to c} \frac{f(x)}{f(x)-g(x)}$

58. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find

a. $\lim_{x\to 4} (g(x)+3)$

b. $\lim_{x\to 4} xf(x)$

c. $\lim_{x\to 4} (g(x))^2$

d. $\lim_{x\to 4} \frac{g(x)}{f(x)-1}$

59. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a. $\lim_{x\to b} (f(x)+g(x))$

b. $\lim_{x\to b} f(x)\cdot g(x)$

c. $\lim_{x\to b} 4g(x)$

d. $\lim_{x\to b} f(x)/g(x)$

60. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Find

a. $\lim_{x\to -2} (p(x)+r(x)+s(x))$

b. $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)$

c. $\lim_{x\to -2} (-4p(x)+5r(x))/s(x)$

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 61–66, evaluate this limit for the given value of x and function f.

61. $f(x) = x^2, \quad x = 1$

62. $f(x) = x^2, \quad x = -2$

63. $f(x) = 3x-4, \quad x = 2$

64. $f(x) = 1/x, \quad x = -2$

65. $f(x) = \sqrt{x}, \quad x = 7$

66. $f(x) = \sqrt{3x+1}, \quad x = 0$

67. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

68. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

69. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

T **b.** Graph $y = 1 - (x^2/6)$, $y = (x \sin x)/(2 - 2\cos x)$, and $y = 1$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

70. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 8.9.) What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T **b.** Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

T You will find a graphing calculator useful for Exercises 71–80.

71. Let $f(x) = (x^2 - 9)/(x + 3)$.

a. Make a table of the values of f at the points $x = -3.1$, -3.01, -3.001, and so on as far as your calculator can go. Then estimate $\lim_{x\to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?

b. Support your conclusions in part (a) by graphing f near $x_0 = -3$ and using Zoom and Trace to estimate y-values on the graph as $x \to -3$.

c. Find $\lim_{x\to -3} f(x)$ algebraically, as in Example 7.

72. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x\to\sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing g near $x_0 = \sqrt{2}$ and using Zoom and Trace to estimate y-values on the graph as $x \to \sqrt{2}$.

c. Find $\lim_{x\to\sqrt{2}} g(x)$ algebraically.

73. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x\to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y-values on the graph as $x \to -6$.

c. Find $\lim_{x\to -6} G(x)$ algebraically.

74. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x\to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing h near $x_0 = 3$ and using Zoom and Trace to estimate y-values on the graph as $x \to 3$.

c. Find $\lim_{x\to 3} h(x)$ algebraically.

75. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a. Make tables of the values of f at values of x that approach $x_0 = -1$ from above and below. Then estimate $\lim_{x\to -1} f(x)$.

b. Support your conclusion in part (a) by graphing f near $x_0 = -1$ and using Zoom and Trace to estimate y-values on the graph as $x \to -1$.

c. Find $\lim_{x\to -1} f(x)$ algebraically.

76. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of F at values of x that approach $x_0 = -2$ from above and below. Then estimate $\lim_{x\to -2} F(x)$.

b. Support your conclusion in part (a) by graphing F near $x_0 = -2$ and using Zoom and Trace to estimate y-values on the graph as $x \to -2$.

c. Find $\lim_{x\to -2} F(x)$ algebraically.

77. Let $g(\theta) = (\sin \theta)/\theta$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta\to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

78. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t\to 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

79. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of f at values of x that approach $x_0 = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $x_0 = 1$.

80. Let $f(x) = (3^x - 1)/x$.

a. Make tables of values of f at values of x that approach $x_0 = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $x_0 = 0$.

81. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x\to c} f(x)$? What can you say about the value of the limit at these points?

82. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x\to 2} f(x) = 0$? Give reasons for your answers.

83. If $\lim_{x\to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x\to 4} f(x)$.

84. If $\lim_{x\to -2} \frac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to -2} f(x)$ b. $\lim_{x\to -2} \frac{f(x)}{x}$

85. a. If $\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 3$, find $\lim_{x\to 2} f(x)$.

b. If $\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x\to 2} f(x)$.

86. If $\lim_{x\to 0} \frac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to 0} f(x)$ b. $\lim_{x\to 0} \frac{f(x)}{x}$

T 87. a. Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T 88. a. Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

In Exercises 89–94, use a CAS to perform the following steps:

a. Plot the function near the point x_0 being approached.

b. From your plot guess the value of the limit.

89. $\lim_{x\to 2} \frac{x^4 - 16}{x - 2}$

90. $\lim_{x\to -1} \frac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

91. $\lim_{x\to 0} \frac{\sqrt[3]{1 + x} - 1}{x}$

92. $\lim_{x\to 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

93. $\lim_{x\to 0} \frac{1 - \cos x}{x \sin x}$

94. $\lim_{x\to 0} \frac{2x^2}{3 - 3\cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. We replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a precise definition, we can prove the limit properties given in the preceding section and establish many important limits.

To show that the limit of $f(x)$ as $x \to x_0$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to x_0. Let us see what this would require if we specified the size of the gap between $f(x)$ and L.

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is clear that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x_0 = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 < 2x - 8 &< 2 \\ 6 < 2x &< 10 \\ 3 < x &< 5 \\ -1 < x - 4 &< 1. \end{aligned}$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Figure 2.15). ■

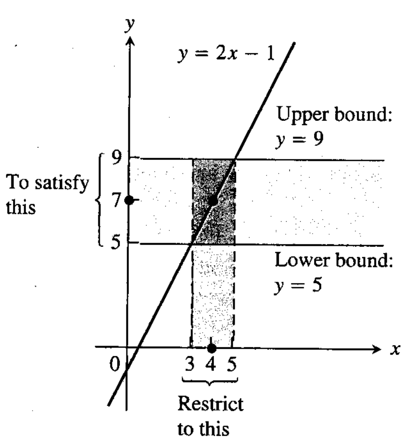


FIGURE 2.15 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).