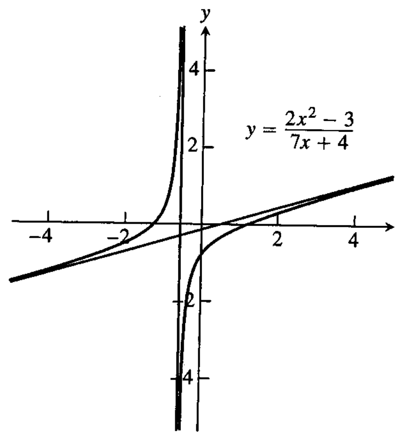


FIGURE 2.41 The function in Example 14 has an oblique asymptote.

EXAMPLE 14 Find the oblique asymptote for the graph of

$$f(x) = \frac{2x^2 - 3}{7x + 4}.$$

Solution By long division, we find

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

$$= \underbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}_{\text{linear function } g(x)} + \underbrace{\frac{-115}{49(7x + 4)}}_{\text{remainder}}$$

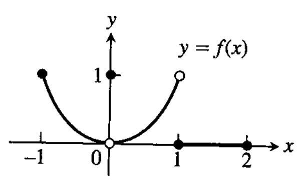
As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the (slanted) line

$$g(x) = \frac{2}{7}x - \frac{8}{49}$$

an asymptote of the graph of f (Figure 2.41). The line $y = g(x)$ is an asymptote both to the right and to the left. In the next section you will see that the function $f(x)$ grows arbitrarily large in absolute value as x approaches $-4/7$, where the denominator becomes zero (Figure 2.41). ■

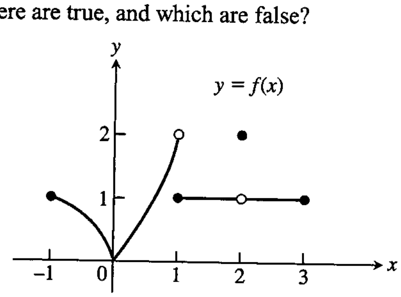
EXERCISES 2.4

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



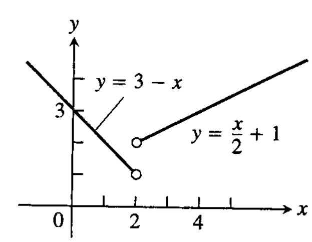
a. $\lim_{x \to -1^+} f(x) = 1$
b. $\lim_{x \to 0^-} f(x) = 0$
c. $\lim_{x \to 0^-} f(x) = 1$
d. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$
e. $\lim_{x \to 0} f(x)$ exists
f. $\lim_{x \to 0} f(x) = 0$
g. $\lim_{x \to 0} f(x) = 1$
h. $\lim_{x \to 1} f(x) = 1$
i. $\lim_{x \to 1} f(x) = 0$
j. $\lim_{x \to 2^-} f(x) = 2$
k. $\lim_{x \to -1^-} f(x)$ does not exist.
l. $\lim_{x \to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



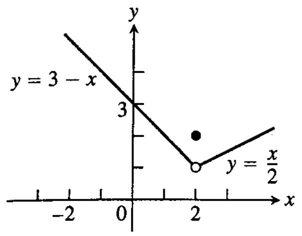
a. $\lim_{x \to -1^+} f(x) = 1$
b. $\lim_{x \to 2} f(x)$ does not exist.
c. $\lim_{x \to 2} f(x) = 2$
d. $\lim_{x \to 1^-} f(x) = 2$
e. $\lim_{x \to 1^+} f(x) = 1$
f. $\lim_{x \to 1} f(x)$ does not exist.
g. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$
h. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(-1, 1)$.
i. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(1, 3)$.
j. $\lim_{x \to -1^-} f(x) = 0$
k. $\lim_{x \to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2. \end{cases}$



a. Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.
b. Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?
c. Find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.
d. Does $\lim_{x \to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



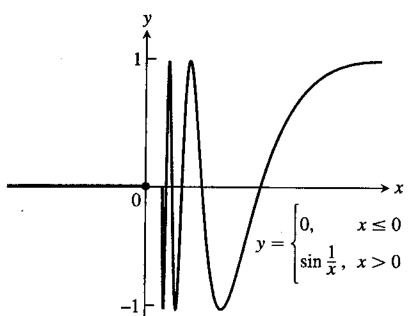
a. Find $\lim_{x\to2^+} f(x)$, $\lim_{x\to2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to-1^-} f(x)$ and $\lim_{x\to-1^+} f(x)$.

d. Does $\lim_{x\to-1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

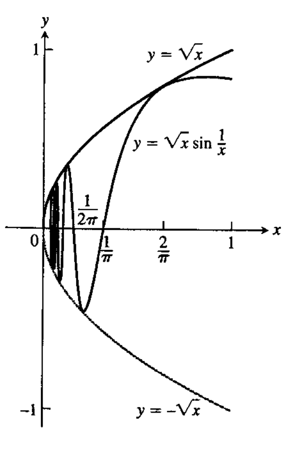


a. Does $\lim_{x\to0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x\to1^-} f(x)$ and $\lim_{x\to1^+} f(x)$.

c. Does $\lim_{x\to1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x\to1^+} f(x)$ and $\lim_{x\to1^-} f(x)$.

c. Does $\lim_{x\to1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c. At what points does only the left-hand limit exist?

d. At what points does only the right-hand limit exist?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Find the limits in Exercises 11–18.

11. $\displaystyle\lim_{x\to-0.5^-} \sqrt{\frac{x+2}{x+1}}$

12. $\displaystyle\lim_{x\to1^+} \sqrt{\frac{x-1}{x+2}}$

13. $\displaystyle\lim_{x\to-2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

14. $\displaystyle\lim_{x\to1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

15. $\displaystyle\lim_{h\to0^+} \frac{\sqrt{h^2+4h+5} - \sqrt{5}}{h}$

16. $\displaystyle\lim_{h\to0^-} \frac{\sqrt{6} - \sqrt{5h^2+11h+6}}{h}$

17. a. $\displaystyle\lim_{x\to-2^+} (x+3)\frac{|x+2|}{x+2}$ **b.** $\displaystyle\lim_{x\to-2^-} (x+3)\frac{|x+2|}{x+2}$

18. a. $\displaystyle\lim_{x\to1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$ **b.** $\displaystyle\lim_{x\to1^-} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.9 in Section 1.1, to help you find the limits in Exercises 19 and 20.

19. a. $\displaystyle\lim_{\theta\to3^+} \frac{\lfloor\theta\rfloor}{\theta}$ **b.** $\displaystyle\lim_{\theta\to3^-} \frac{\lfloor\theta\rfloor}{\theta}$

20. a. $\displaystyle\lim_{t\to4^+} (t - \lfloor t \rfloor)$ **b.** $\displaystyle\lim_{t\to4^-} (t - \lfloor t \rfloor)$

Find the limits in Exercises 21–36.

21. $\displaystyle\lim_{\theta\to0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\displaystyle\lim_{t\to0} \frac{\sin kt}{t}$ (k constant)

23. $\lim_{y\to 0} \frac{\sin 3y}{4y}$

24. $\lim_{h\to 0^-} \frac{h}{\sin 3h}$

25. $\lim_{x\to 0} \frac{\tan 2x}{x}$

26. $\lim_{t\to 0} \frac{2t}{\tan t}$

27. $\lim_{x\to 0} \frac{x \csc 2x}{\cos 5x}$

28. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

29. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x \cos x}$

30. $\lim_{x\to 0} \frac{x^2 - x + \sin x}{2x}$

31. $\lim_{t\to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

32. $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h}$

33. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$

34. $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$

35. $\lim_{x\to 0} \frac{\tan 3x}{\sin 8x}$

36. $\lim_{y\to 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

In Exercises 37–42, find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

37. $f(x) = \frac{2}{x} - 3$

38. $f(x) = \pi - \frac{2}{x^2}$

39. $g(x) = \frac{1}{2 + (1/x)}$

40. $g(x) = \frac{1}{8 - (5/x^2)}$

41. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$

42. $h(x) = \frac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 43–50.

43. $\lim_{x\to\infty} \frac{\sin 2x}{x}$

44. $\lim_{\theta\to -\infty} \frac{\cos\theta}{3\theta}$

45. $\lim_{t\to -\infty} \frac{2 - t + \sin t}{t + \cos t}$

46. $\lim_{r\to\infty} \frac{r + \sin r}{2r + 7 - 5\sin r}$

47. $\lim_{x\to\infty} e^{-x}\sin x$

48. $\lim_{x\to -\infty} e^x \cos^{-1}\left(\frac{1}{x}\right)$

49. $\lim_{x\to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

50. $\lim_{x\to\infty} \frac{3x^2 + e^{-x}}{\sin(1/x) - 2x^2}$

In Exercises 51–60, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

51. $f(x) = \frac{2x + 3}{5x + 7}$

52. $f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7}$

53. $f(x) = \frac{x + 1}{x^2 + 3}$

54. $f(x) = \frac{3x + 7}{x^2 - 2}$

55. $h(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$

56. $g(x) = \frac{1}{x^3 - 4x + 1}$

57. $g(x) = \frac{10x^5 + x^4 + 31}{x^6}$

58. $h(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

59. $h(x) = \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

60. $h(x) = \frac{-x^4}{x^4 - 7x^3 + 7x^2 + 9}$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 61–66.

61. $\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

62. $\lim_{x\to\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

63. $\lim_{x\to -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

64. $\lim_{x\to\infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

65. $\lim_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

66. $\lim_{x\to -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

67. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

68. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

69. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

70. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

71. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x\to\infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x\to -\infty} (f(x)/g(x))$? Give reasons for your answer.

72. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

73. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

74. Find $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - x}\right)$.

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 75 and 76.

75. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

76. If f has the constant value $f(x) = k$, then $\lim_{x\to -\infty} f(x) = k$.

77. Given $\epsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x - 5} < \epsilon$. What limit is being verified and what is its value?

78. Given $\epsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4 - x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 79 and 80.

79. $\lim_{x\to 0^-} \frac{x}{|x|} = -1$

80. $\lim_{x\to 2^+} \frac{x - 2}{|x - 2|} = 1$

81. **Greatest integer function** Find **(a)** $\lim_{x\to 400^+} \lfloor x \rfloor$ and **(b)** $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can anything be said about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answers.

82. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find **(a)** $\lim_{x\to 0^+} f(x)$ and **(b)** $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can anything be said about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

T Sometimes a change of variable can change an unfamiliar expression into one whose limit we know how to find, such as in Example 11, where we substituted $t = 1/x$ when finding a limit as $x \to \infty$. This suggests a creative way to "see" limits at infinity. Describe the procedure and use it to picture and determine limits in Exercises 83–88.

83. $\lim_{x\to\pm\infty} x \sin \frac{1}{x}$

84. $\lim_{x\to-\infty} \frac{\cos(1/x)}{1 + (1/x)}$

85. $\lim_{x\to\pm\infty} \frac{3x + 4}{2x - 5}$

86. $\lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x}$

87. $\lim_{x\to\pm\infty} \left(3 + \frac{2}{x}\right)\left(\cos\frac{1}{x}\right)$

88. $\lim_{x\to\infty} \left(\frac{3}{x^2} - \cos\frac{1}{x}\right)\left(1 + \sin\frac{1}{x}\right)$

2.5 Infinite Limits and Vertical Asymptotes

In this section we extend the concept of limit to *infinite limits*, which are not limits as before, but rather an entirely new use of the term limit. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large. We continue our analysis of graphs of rational functions from the last section, using vertical asymptotes for numerically large values of x.

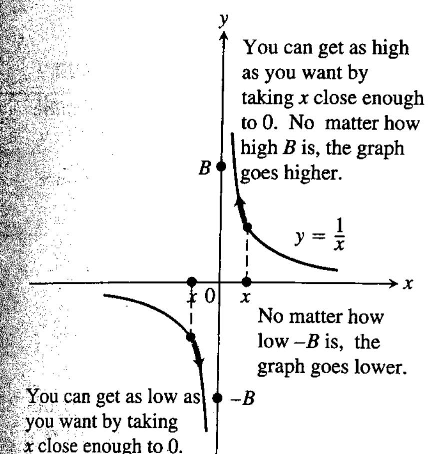


FIGURE 2.42 One-sided infinite limits: $\lim_{x\to0^+} \frac{1}{x} = \infty$ and $\lim_{x\to0^-} \frac{1}{x} = -\infty$

Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.42). Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{x} = \infty.$$

In writing this, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x\to0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$.*

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.42.) We write

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative.*

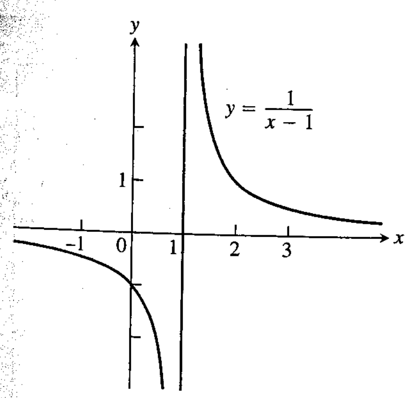


FIGURE 2.43 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 1).

EXAMPLE 1 Find $\lim_{x\to1^+} \frac{1}{x-1}$ and $\lim_{x\to1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.43). Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x\to1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x - 1) \to 0^+$ and $1/(x - 1) \to \infty$. As $x \to 1^-$, we have $(x - 1) \to 0^-$ and $1/(x - 1) \to -\infty$. ■

EXAMPLE 2 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{near} \quad x = 0.$$