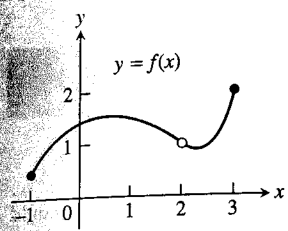
EXERCISES 2.6

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

1.



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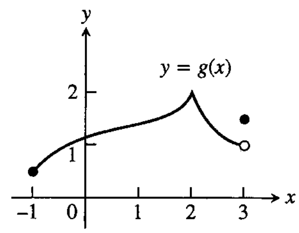

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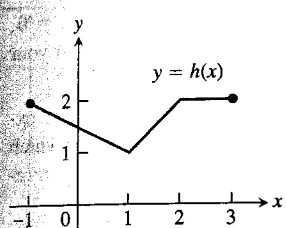

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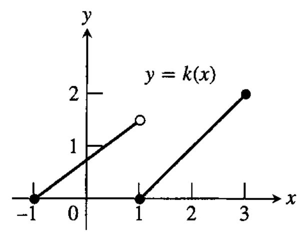
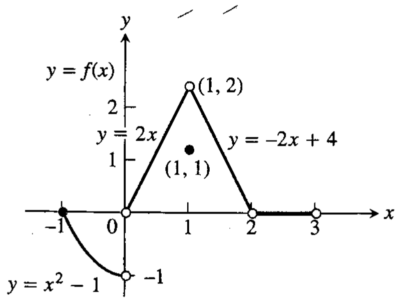


Exercises 5–10 are about the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. a. Does $f(-1)$ exist?
   b. Does $\lim_{x\to -1^+} f(x)$ exist?
   c. Does $\lim_{x\to -1^+} f(x) = f(-1)$?
   d. Is f continuous at $x = -1$?
6. a. Does $f(1)$ exist?
   b. Does $\lim_{x\to 1} f(x)$ exist?
   c. Does $\lim_{x\to 1} f(x) = f(1)$?
   d. Is f continuous at $x = 1$?
7. a. Is f defined at $x = 2$? (Look at the definition of f.)
   b. Is f continuous at $x = 2$?
8. At what values of x is f continuous?
9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?
10. To what new value should $f(1)$ be changed to remove the discontinuity?

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. Exercise 1, Section 2.4
12. Exercise 2, Section 2.4

At what points are the functions in Exercises 13–28 continuous?

13. $y = \dfrac{1}{x-2} - 3x$
14. $y = \dfrac{1}{(x+2)^2} + 4$
15. $y = \dfrac{x+1}{x^2 - 4x + 3}$
16. $y = \dfrac{x+3}{x^2 - 3x - 10}$
17. $y = |x - 1| + \sin x$
18. $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$
19. $y = \dfrac{\cos x}{x}$
20. $y = \dfrac{x+2}{\cos x}$
21. $y = \csc 2x$
22. $y = \tan \dfrac{\pi x}{2}$
23. $y = \dfrac{x \tan x}{x^2 + 1}$
24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$
25. $y = \sqrt{2x + 3}$
26. $y = \sqrt[4]{3x - 1}$
27. $y = (2x - 1)^{1/3}$
28. $y = (2 - x)^{1/5}$

Find the limits in Exercises 29–34. Are the functions continuous at the point being approached?

29. $\lim_{x\to\pi} \sin(x - \sin x)$
30. $\lim_{t\to 0} \sin\left(\dfrac{\pi}{2}\cos(\tan t)\right)$
31. $\lim_{y\to 1} \sec(y \sec^2 y - \tan^2 y - 1)$
32. $\lim_{x\to 0} \tan\left(\dfrac{\pi}{4}\cos(\sin x^{1/3})\right)$
33. $\lim_{x\to 0^+} \sin\left(\dfrac{\pi}{2} e^{\sqrt{x}}\right)$
34. $\lim_{x\to 1} \cos^{-1}(\ln \sqrt{x})$
35. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.
36. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.
37. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

38. Define $g(4)$ in a way that extends
$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$
to be continuous at $x = 4$.

39. For what value of a is
$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$
continuous at every x?

40. For what value of b is
$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$
continuous at every x?

T In Exercises 41–44, graph the function f to see whether it appears to have a continuous extension to the origin. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

41. $f(x) = \dfrac{10^x - 1}{x}$

42. $f(x) = \dfrac{10^{|x|} - 1}{x}$

43. $f(x) = \dfrac{\sin x}{|x|}$

44. $f(x) = (1 + 2x)^{1/x}$

45. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

46. Explain why the equation $\cos x = x$ has at least one solution.

47. Roots of a cubic Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

48. A function value Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x.

49. Solving an equation If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals **(a)** π; **(b)** $-\sqrt{3}$; **(c)** 5,000,000.

50. Explain why the following five statements ask for the same information.

a. Find the roots of $f(x) = x^3 - 3x - 1$.

b. Find the x-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

c. Find all the values of x for which $x^3 - 3x = 1$.

d. Find the x-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

e. Solve the equation $x^3 - 3x - 1 = 0$.

51. Removable discontinuity Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

52. Nonremovable discontinuity Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

53. A function discontinuous at every point

a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function
$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$
is discontinuous at every point.

b. Is f right-continuous or left-continuous at any point?

54. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

55. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

56. Discontinuous composite of continuous functions Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 10? Give reasons for your answer.

57. Never-zero continuous functions Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

58. Stretching a rubber band Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

59. A fixed point theorem Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

60. The sign-preserving property of continuous functions Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$. Notice how remarkable this conclusion is. Although f is defined throughout (a, b), it is not required to be continuous at any point except c. That and the condition $f(c) \neq 0$ are enough to make f different from zero (positive or negative) throughout an entire interval.

61. Prove that f is continuous at c if and only if
$$\lim_{h \to 0} f(c + h) = f(c).$$

62. Use Exercise 61 together with the identities
$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$
$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$
to prove that $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

T Use a graphing calculator or computer grapher to solve the equations in Exercises 63–70.

63. $x^3 - 3x - 1 = 0$

64. $2x^3 - 2x^2 - 2x + 1 = 0$

65. $x(x-1)^2 = 1$ (one root)

66. $x^x = 2$

67. $\sqrt{x} + \sqrt{1+x} = 4$

68. $x^3 - 15x + 1 = 0$ (three roots)

69. $\cos x = x$ (one root). Make sure you are using radian mode.

70. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

2.7 Tangents and Derivatives at a Point

This section makes precise our discussion in Section 2.1 of slopes and tangent lines to a curve. We saw how these notions are related to the instantaneous rate of change of a function, which is an interpretation of the *derivative* of the function. We define the derivative here.

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.67). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

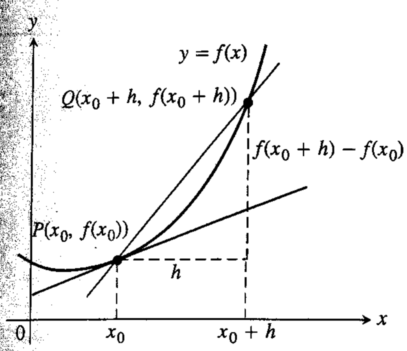


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at P. Let's look at another example.

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at $x = a \neq 0$.

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-h}{ha(a+h)} = \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$