1.1 FUNCTIONS AND CHANGE

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. Syracuse, New York has the highest annual snowfall of any US city because of the "lake effect" snow coming from cold Northwest winds blowing over nearby Lake Erie. Lake effect snowfall has been heavier over the last few decades; some have suggested this is due to the warming of Lake Erie by climate change. In December 2010, Syracuse got 66.9 inches of snow in one 12 day period, all of it from lake effect snow. See Table 1.1.

Table 1.1 *Daily snowfall in Syracuse, December 5–16, 2010*

Date (December 2010)	5	6	7	8	9	10	11	12	13	14	15	16
Snowfall in inches	6.8	12.2	9.3	14.9	1.9	0.1	0.0	0.0	1.4	5.0	11.9	3.4

You may not have thought of something so unpredictable as daily snowfall as being a function, but it *is* a function of date, because each day gives rise to one snowfall total. There is no formula for the daily snowfall (otherwise we would not need a weather bureau), but nevertheless the daily snowfall in Syracuse does satisfy the definition of a function: Each date, t, has a unique snowfall, S, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the snowfall example, the domain is the set of December dates $\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and the range is the set of daily snowfalls $\{0.0, 0.1, 1.4, 1.9, 3.4, 5.0, 6.8, 9.3, 11.9, 12.2, 14.9\}$. We call the function f and write $S = f(t)$. Notice that a function may have identical outputs for different inputs (December 11 and 12, for example).

Some quantities, such as date, are *discrete*, meaning they take only certain isolated values (dates must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \le t \le b$ is called a *closed interval* and written $[a, b]$.

The set of numbers t such that $a < t < b$ is called an *open interval* and written (a, b).

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily snowfall in Syracuse can be represented by the graph in Figure 1.1, as well as by Table 1.1.

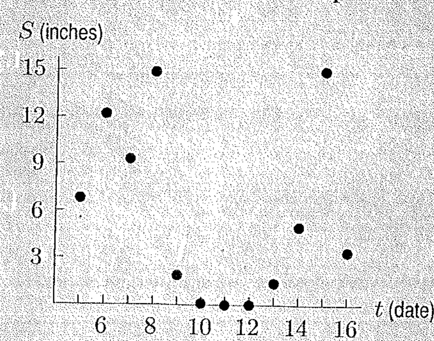


Figure 1.1: Syracuse snowfall, December, 2010

As another example of a function, consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed by the formula

$$C = 4T - 160$$

to a fair degree of accuracy. We write $C = f(T)$ to express the fact that we think of C as a function of T and that we have named this function f. The graph of this function is in Figure 1.2.

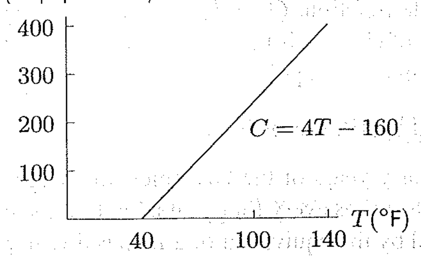


Figure 1.2: Cricket chirp rate versus temperature

Examples of Domain and Range

If the domain of a function is not specified, we usually take it to be the largest possible set of real numbers. For example, we usually think of the domain of the function $f(x) = x^2$ as all real numbers. However, the domain of the function $g(x) = 1/x$ is all real numbers except zero, since we cannot divide by zero.

Sometimes we restrict the domain to be smaller than the largest possible set of real numbers. For example, if the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to nonnegative values of x.

Example 1 The function $C = f(T)$ gives chirp rate as a function of temperature. We restrict this function to temperatures for which the predicted chirp rate is positive, and up to the highest temperature ever recorded at a weather station, 136°F. What is the domain of this function f?

Solution If we consider the equation

$$C = 4T - 160$$

simply as a mathematical relationship between two variables C and T, any T value is possible. However, if we think of it as a relationship between cricket chirps and temperature, then C cannot be less than 0. Since $C = 0$ leads to $0 = 4T - 160$, and so $T = 40$°F, we see that T cannot be less than 40°F. (See Figure 1.2.) In addition, we are told that the function is not defined for temperatures above 136°. Thus, for the function $C = f(T)$ we have

$$\begin{aligned}\text{Domain} &= \text{All } T \text{ values between } 40°\text{F and } 136°\text{F}\\ &= \text{All } T \text{ values with } 40 \leq T \leq 136\\ &= [40, 136].\end{aligned}$$

Example 2 Find the range of the function f, given the domain from Example 1. In other words, find all possible values of the chirp rate, C, in the equation $C = f(T)$.

Solution Again, if we consider $C = 4T - 160$ simply as a mathematical relationship, its range is all real C values. However, when thinking of the meaning of $C = f(T)$ for crickets, we see that the function predicts cricket chirps per minute between 0 (at $T = 40$°F) and 384 (at $T = 136$°F). Hence,

$$\begin{aligned}\text{Range} &= \text{All } C \text{ values from 0 to 384}\\ &= \text{All } C \text{ values with } 0 \leq C \leq 384\\ &= [0, 384].\end{aligned}$$

In using the temperature to predict the chirp rate, we thought of the temperature as the *independent variable* and the chirp rate as the *dependent variable*. However, we could do this backward, and calculate the temperature from the chirp rate. From this point of view, the temperature is dependent on the chirp rate. Thus, which variable is dependent and which is independent may depend on your viewpoint.

Linear Functions

The chirp-rate function, $C = f(T)$, is an example of a *linear function*. A function is linear if its slope, or rate of change, is the same at every point. The rate of change of a function that is not linear may vary from point to point.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 8 inches every four years. Table 1.2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches a year. So the height was a linear function of time from 1900 to 1912. If y is the winning height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

Since $y = f(t)$ increases with t, we say that f is an *increasing function*. The coefficient 2 tells us the rate, in inches per year, at which the height increases.

Table 1.2 *Men's Olympic pole vault winning height (approximate)*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

This rate of increase is the *slope* of the line in Figure 1.3. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches/year.}$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

What about the constant 130? This represents the initial height in 1900, when $t = 0$. Geometrically, 130 is the *intercept* on the vertical axis.

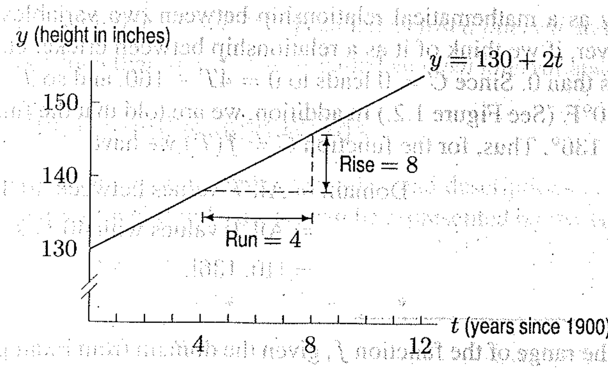


Figure 1.3: Olympic pole vault records

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it doesn't exactly. The formula $y = 130+2t$ predicts that the height in the 2008 Olympics would be 346 inches or 28 feet 10 inches, which is considerably higher than the actual value of 19 feet 6.65 inches. There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.2 is discrete, because it is given only at specific points (every four years). However, we have treated the variable t as though it were continuous, because the function $y = 130 + 2t$ makes sense for all values of t. The graph in Figure 1.3 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

As the pole vault heights have increased over the years, the time to run the mile has decreased. If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.39t.$$

The 260 tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.39, tells us that the world record decreased by about 0.39 seconds per year. We say that g is a *decreasing function*.

Difference Quotients and Delta Notation

We use the symbol Δ (the Greek letter capital delta) to mean "change in," so Δx means change in x and Δy means change in y.

The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2, using the formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.4.) Since $m = \Delta y/\Delta x$, the units of m are y-units over x-units.

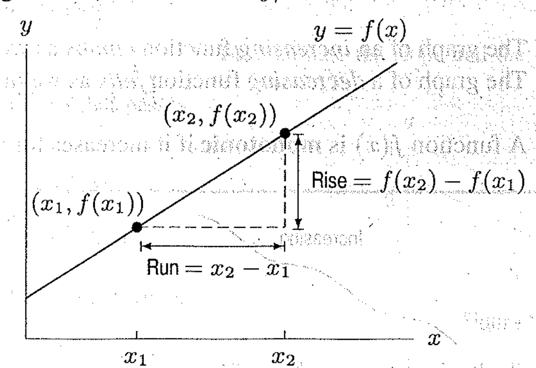


Figure 1.4: Difference quotient $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Families of Linear Functions

A **linear function** has the form

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x.
- b is the **vertical intercept**, or value of y when x is zero.

Notice that if the slope, m, is zero, we have $y = b$, a horizontal line.

To recognize that a table of x and y values comes from a linear function, $y = b + mx$, look for differences in y-values that are constant for equally spaced x-values.

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, give a *family of functions*. All the functions in a family share certain properties—in this case, all the graphs are straight lines. The constants m and b are called *parameters*; their meaning is shown in Figures 1.5 and 1.6. Notice that the greater the magnitude of m, the steeper the line.

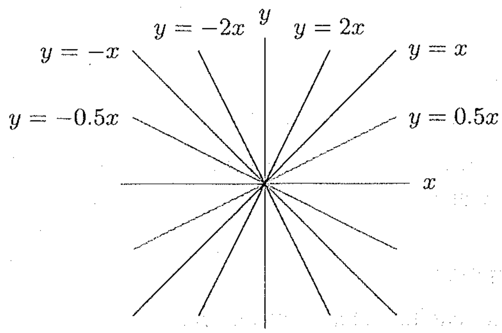


Figure 1.5: The family $y = mx$ (with $b = 0$)

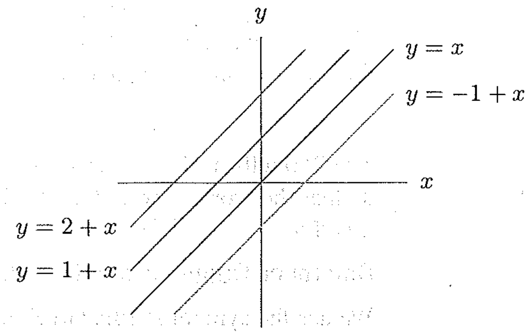


Figure 1.6: The family $y = b + x$ (with $m = 1$)

Increasing versus Decreasing Functions

The terms increasing and decreasing can be applied to other functions, not just linear ones. See Figure 1.7. In general,

> A function f is **increasing** if the values of $f(x)$ increase as x increases.
> A function f is **decreasing** if the values of $f(x)$ decrease as x increases.
>
> The graph of an *increasing* function *climbs* as we move from left to right.
> The graph of a *decreasing* function *falls* as we move from left to right.
>
> A function $f(x)$ is **monotonic** if it increases for all x or decreases for all x.

Figure 1.7: Increasing and decreasing functions

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, the area, A, of a circle is proportional to the square of the radius, r, because

$$A = f(r) = \pi r^2.$$

> We say y is (directly) **proportional** to x if there is a nonzero constant k such that
>
> $$y = kx.$$
>
> This k is called the constant of proportionality.

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v, at which you make a 50-mile trip is inversely proportional to the time, t, taken, because v is proportional to $1/t$:

$$v = 50\left(\frac{1}{t}\right) = \frac{50}{t}.$$

Exercises and Problems for Section 1.1

Exercises

1. The population of a city, P, in millions, is a function of t, the number of years since 1970, so $P = f(t)$. Explain the meaning of the statement $f(35) = 12$ in terms of the population of this city.

2. The pollutant PCB (polychlorinated biphenyl) affects the thickness of pelican eggs. Thinking of the thickness, T, of the eggs, in mm, as a function of the concentration, P, of PCBs in ppm (parts per million), we have $T = f(P)$. Explain the meaning of $f(200)$ in terms of thickness of pelican eggs and concentration of PCBs.

3. Describe what Figure 1.8 tells you about an assembly line whose productivity is represented as a function of the number of workers on the line.

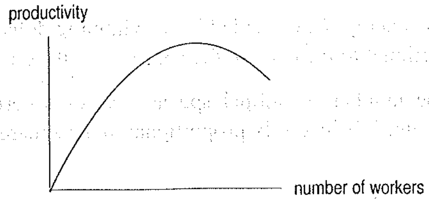


Figure 1.8

For Exercises 4–7, find an equation for the line that passes through the given points.

4. $(0,0)$ and $(1,1)$
5. $(0,2)$ and $(2,3)$
6. $(-2,1)$ and $(2,3)$
7. $(-1,0)$ and $(2,6)$

For Exercises 8–11, determine the slope and the y-intercept of the line whose equation is given.

8. $2y + 5x - 8 = 0$
9. $7y + 12x - 2 = 0$
10. $-4y + 2x + 8 = 0$
11. $12x = 6y + 4$

12. Match the graphs in Figure 1.9 with the following equations. (Note that the x and y scales may be unequal.)

(a) $y = x - 5$ (b) $-3x + 4 = y$
(c) $5 = y$ (d) $y = -4x - 5$
(e) $y = x + 6$ (f) $y = x/2$

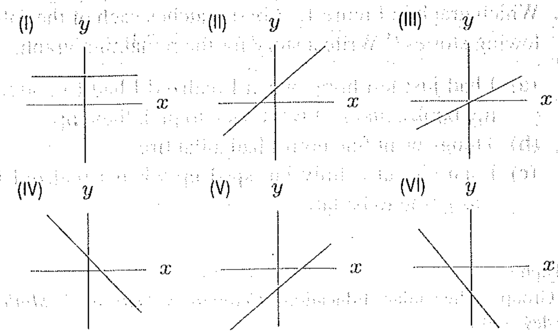


Figure 1.9

13. Match the graphs in Figure 1.10 with the following equations. (Note that the x and y scales may be unequal.)

(a) $y = -2.72x$ (b) $y = 0.01 + 0.001x$
(c) $y = 27.9 - 0.1x$ (d) $y = 0.1x - 27.9$
(e) $y = -5.7 - 200x$ (f) $y = x/3.14$

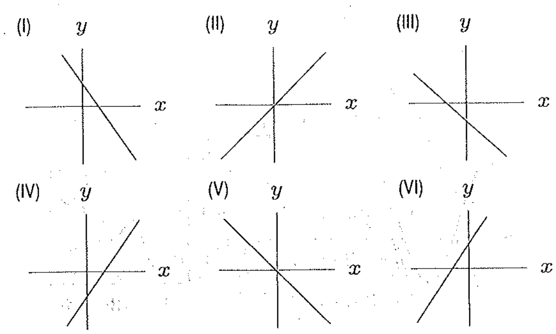


Figure 1.10

14. Estimate the slope and the equation of the line in Figure 1.11.

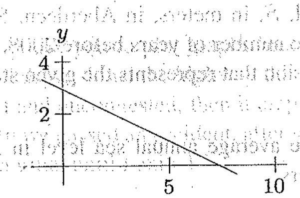


Figure 1.11

15. Find an equation for the line with slope m through the point (a, c).

16. Find a linear function that generates the values in Table 1.3.

Table 1.3

x	5.2	5.3	5.4	5.5	5.6
y	27.8	29.2	30.6	32.0	33.4

For Exercises 17–19, use the facts that parallel lines have equal slopes and that the slopes of perpendicular lines are negative reciprocals of one another.

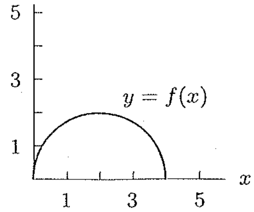
17. Find an equation for the line through the point $(2, 1)$ which is perpendicular to the line $y = 5x - 3$.

18. Find equations for the lines through the point $(1, 5)$ that are parallel to and perpendicular to the line with equation $y + 4x = 7$.

19. Find equations for the lines through the point (a, b) that are parallel and perpendicular to the line $y = mx + c$, assuming $m \neq 0$.

For Exercises 20–23, give the approximate domain and range of each function. Assume the entire graph is shown.

20.



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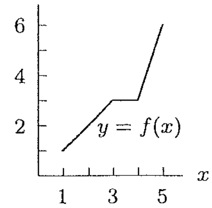

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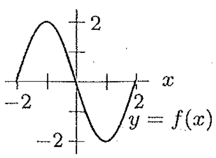

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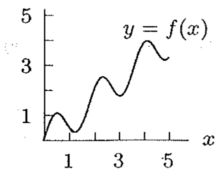


Find domain and range in Exercises 24–25.

24. $y = x^2 + 2$

25. $y = \dfrac{1}{x^2 + 2}$

26. If $f(t) = \sqrt{t^2 - 16}$, find all values of t for which $f(t)$ is a real number. Solve $f(t) = 3$.

In Exercises 27–31, write a formula representing the function.

27. The volume of a sphere is proportional to the cube of its radius, r.

28. The average velocity, v, for a trip over a fixed distance, d, is inversely proportional to the time of travel, t.

29. The strength, S, of a beam is proportional to the square of its thickness, h.

30. The energy, E, expended by a swimming dolphin is proportional to the cube of the speed, v, of the dolphin.

31. The number of animal species, N, of a certain body length, l, is inversely proportional to the square of l.

Problems

In Problems 32–35 the function $S = f(t)$ gives the average annual sea level, S, in meters, in Aberdeen, Scotland,[1] as a function of t, the number of years before 2008. Write a mathematical expression that represents the given statement.

32. In 1983 the average annual sea level in Aberdeen was 7.019 meters.

33. The average annual sea level in Aberdeen in 2008.

34. The average annual sea level in Aberdeen was the same in 1865 and 1911.

35. The average annual sea level in Aberdeen increased by 1 millimeter from 2007 to 2008.

36. In December 2010, the snowfall in Minneapolis was unusually high,[2] leading to the collapse of the roof of the Metrodome. Figure 1.12 gives the snowfall, S, in Minneapolis for December 6–15, 2010.

(a) How do you know that the snowfall data represents a function of date?
(b) Estimate the snowfall on December 12.
(c) On which day was the snowfall more than 10 inches?
(d) During which consecutive two-day interval was the increase in snowfall largest?

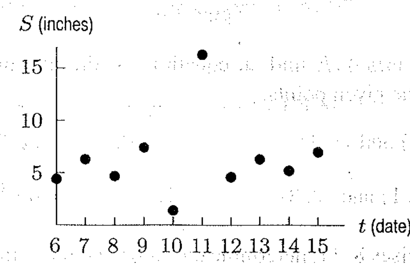


Figure 1.12

37. The value of a car, $V = f(a)$, in thousands of dollars, is a function of the age of the car, a, in years.

(a) Interpret the statement $f(5) = 6$.
(b) Sketch a possible graph of V against a. Is f an increasing or decreasing function? Explain.
(c) Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.

38. Which graph in Figure 1.13 best matches each of the following stories?[3] Write a story for the remaining graph.

(a) I had just left home when I realized I had forgotten my books, and so I went back to pick them up.
(b) Things went fine until I had a flat tire.
(c) I started out calmly but sped up when I realized I was going to be late.

[1] www.decc.gov.uk, accessed June 2011
[2] http://www.crh.noaa.gov/mpx/Climate/DisplayRecords.php
[3] Adapted from Jan Terwel, "Real Math in Cooperative Groups in Secondary Education." *Cooperative Learning in Mathematics*, ed. Neal Davidson, p. 234 (Reading: Addison Wesley, 1990).

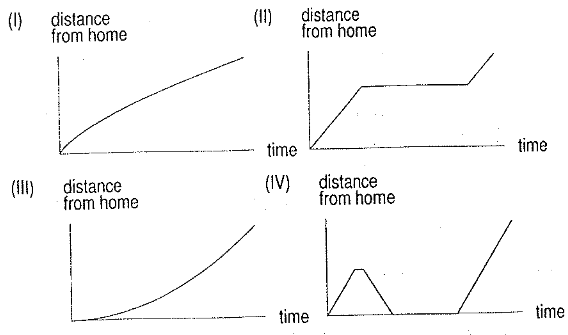


Figure 1.13

39. An object is put outside on a cold day at time $t = 0$. Its temperature, $H = f(t)$, in °C, is graphed in Figure 1.14.

(a) What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.

(b) Explain what the vertical intercept, a, and the horizontal intercept, b, represent in terms of temperature of the object and time outside.

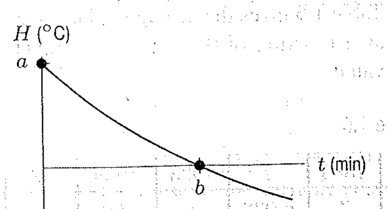


Figure 1.14

40. A rock is dropped from a window and falls to the ground below. The height, s (in meters), of the rock above ground is a function of the time, t (in seconds), since the rock was dropped, so $s = f(t)$.

(a) Sketch a possible graph of s as a function of t.

(b) Explain what the statement $f(7) = 12$ tells us about the rock's fall.

(c) The graph drawn as the answer for part (a) should have a horizontal and vertical intercept. Interpret each intercept in terms of the rock's fall.

41. In a California town, the monthly charge for waste collection is \$8 for 32 gallons of waste and \$12.32 for 68 gallons of waste.

(a) Find a linear formula for the cost, C, of waste collection as a function of the number of gallons of waste, w.

(b) What is the slope of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.

(c) What is the vertical intercept of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.

42. For tax purposes, you may have to report the value of your assets, such as cars or refrigerators. The value you report drops with time. "Straight-line depreciation" assumes that the value is a linear function of time. If a \$950 refrigerator depreciates completely in seven years, find a formula for its value as a function of time.

43. A company rents cars at \$40 a day and 15 cents a mile. Its competitor's cars are \$50 a day and 10 cents a mile.

(a) For each company, give a formula for the cost of renting a car for a day as a function of the distance traveled.

(b) On the same axes, graph both functions.

(c) How should you decide which company is cheaper?

44. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount monthly plus a charge for each cubic foot of water used. A household using 1000 cubic feet was billed \$40, while one using 1600 cubic feet was billed \$55.

(a) What is the charge per cubic foot?

(b) Write an equation for the total cost of a resident's water as a function of cubic feet of water used.

(c) How many cubic feet of water used would lead to a bill of \$100?

Problems 45–48 ask you to plot graphs based on the following story: "As I drove down the highway this morning, at first traffic was fast and uncongested, then it crept nearly bumper-to-bumper until we passed an accident, after which traffic flow went back to normal until I exited."

45. Driving speed against time on the highway

46. Distance driven against time on the highway

47. Distance from my exit vs time on the highway

48. Distance between cars vs distance driven on the highway

49. Let $f(t)$ be the number of US billionaires in the US in year t.

(a) Express the following statements[4] in terms of f.

(i) In 1985 there were 13 US billionaires.

(ii) In 1990 there were 99 US billionaires.

(b) Find the average yearly increase in the number of US billionaires between 1985 and 1990. Express this using f.

(c) Assuming the yearly increase remains constant, find a formula predicting the number of US billionaires in year t.

[4] //hypertextbook.com/facts/2005/MichelleLee.shtml

50. An alternative to petroleum-based diesel fuel, biodiesel, is derived from renewable resources such as food crops, algae, and animal oils. The table shows the recent annual percent growth in US biodiesel consumption.[5]

Year	2005	2006	2007	2008	2009
% growth over previous yr	237	186.6	37.2	−11.7	7.3

(a) Find the largest time interval over which the percentage growth in the US consumption of biodiesel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.

(b) Find the largest time interval over which the actual US consumption of biodiesel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.

51. Hydroelectric power is electric power generated by the force of moving water. Figure 1.15 shows[6] the annual percent growth in hydroelectric power consumption by the US industrial sector between 2004 and 2009.

(a) Find the largest time interval over which the percentage growth in the US consumption of hydroelectric power was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.

(b) Find the largest time interval over which the actual US consumption of hydroelectric power was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.

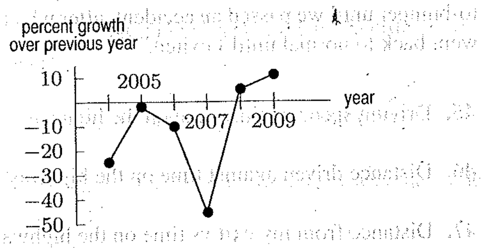


Figure 1.15

52. Solar panels are arrays of photovoltaic cells that convert solar radiation into electricity. The table shows the annual percent change in the US price per watt of a solar panel.[7]

Year	2004	2005	2006	2007	2008
% growth over previous yr	−5.7	6.7	9.7	−3.7	3.6

(a) Find the largest time interval over which the percentage growth in the US price per watt of a solar panel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.

(b) Find the largest time interval over which the actual price per watt of a solar panel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.

53. Table 1.4 shows the average annual sea level, S, in meters, in Aberdeen, Scotland,[8] as a function of time, t, measured in years before 2008.

Table 1.4

t	0	25	50	75	100	125
S	7.094	7.019	6.992	6.965	6.938	6.957

(a) What was the average sea level in Aberdeen in 2008?

(b) In what year was the average sea level 7.019 meters? 6.957 meters?

(c) Table 1.5 gives the average sea level, S, in Aberdeen as a function of the year, x. Complete the missing values.

Table 1.5

x	1883	?	1933	1958	1983	2008
S	?	6.938	?	6.992	?	?

54. A controversial 1992 Danish study[9] reported that men's average sperm count has decreased from 113 million per milliliter in 1940 to 66 million per milliliter in 1990.

(a) Express the average sperm count, S, as a linear function of the number of years, t, since 1940.

(b) A man's fertility is affected if his sperm count drops below about 20 million per milliliter. If the linear model found in part (a) is accurate, in what year will the average male sperm count fall below this level?

55. The table gives the average weight, w, in pounds, of American men in their sixties for height, h, in inches.[10]

(a) How do you know that the data in this table could represent a linear function?

(b) Find weight, w, as a linear function of height, h. What is the slope of the line? What are the units for the slope?

[5] http://www.eia.doe.gov/aer/renew.html. Accessed February 2011.
[6] Yearly values have been joined with segments to highlight trends in the data, however values in between years should not be inferred from the segments. From http://www.eia.doe.gov/aer/renew.html. Accessed February 2011.
[7] We use the official price per peak watt, which uses the maximum number of watts a solar panel can produce under ideal conditions. From http://www.eia.doe.gov/aer/renew.html. Accessed February 2011.
[8] www.decc.gov.uk, accessed June 2011.
[9] "Investigating the Next Silent Spring," *US News and World Report*, pp. 50–52 (March 11, 1996).
[10] Adapted from "Average Weight of Americans by Height and Age," *The World Almanac* (New Jersey: Funk and Wagnalls, 1992), p. 956.

(c) Find height, h, as a linear function of weight, w. What is the slope of the line? What are the units for the slope?

h (inches)	68	69	70	71	72	73	74	75
w (pounds)	166	171	176	181	186	191	196	201

56. An airplane uses a fixed amount of fuel for takeoff, a (different) fixed amount for landing, and a third fixed amount per mile when it is in the air. How does the total quantity of fuel required depend on the length of the trip? Write a formula for the function involved. Explain the meaning of the constants in your formula.

57. The cost of planting seed is usually a function of the number of acres sown. The cost of the equipment is a *fixed cost* because it must be paid regardless of the number of acres planted. The costs of supplies and labor vary with the number of acres planted and are called *variable costs*. Suppose the fixed costs are \$10,000 and the variable costs are \$200 per acre. Let C be the total cost, measured in thousands of dollars, and let x be the number of acres planted.

(a) Find a formula for C as a function of x.
(b) Graph C against x.
(c) Which feature of the graph represents the fixed costs? Which represents the variable costs?

58. You drive at a constant speed from Chicago to Detroit, a distance of 275 miles. About 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.

59. (a) Consider the functions graphed in Figure 1.16(a). Find the coordinates of C.

(b) Consider the functions in Figure 1.16(b). Find the coordinates of C in terms of b.

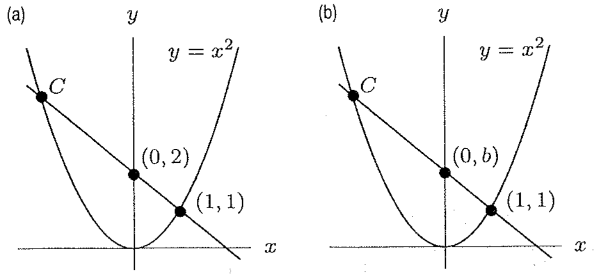


Figure 1.16

60. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen. What do the experimental data in Table 1.6 tell you about Galileo's hypothesis? What alternative hypothesis is suggested by the two sets of data in Table 1.6 and Table 1.7?

Table 1.6

Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

Table 1.7

Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

Strengthen Your Understanding

In Problems 61–62, explain what is wrong with the statement.

61. Values of y on the graph of $y = 0.5x - 3$ increase more slowly than values of y on the graph of $y = 0.5 - 3x$.

62. The equation $y = 2x + 1$ indicates that y is directly proportional to x with a constant of proportionality 2.

In Problems 63–64, give an example of:

63. A linear function with a positive slope and a negative x-intercept.

64. A formula representing the statement "q is inversely proportional to the cube root of p and has a positive constant of proportionality."

Are the statements in Problems 65–68 true or false? Give an explanation for your answer.

65. For any two points in the plane, there is a linear function whose graph passes through them.

66. If $y = f(x)$ is a linear function, then increasing x by 1 unit changes the corresponding y by m units, where m is the slope.

67. If y is a linear function of x, then the ratio y/x is constant for all points on the graph at which $x \neq 0$.

68. If $y = f(x)$ is a linear function, then increasing x by 2 units adds $m + 2$ units to the corresponding y, where m is the slope.

69. Which of the following functions has its domain identical with its range?

(a) $f(x) = x^2$ (b) $g(x) = \sqrt{x}$
(c) $h(x) = x^3$ (d) $i(x) = |x|$

1.2 EXPONENTIAL FUNCTIONS

Population Growth

The population of Burkina Faso, a sub-Saharan African country,[11] from 2003 to 2009 is given in Table 1.8. To see how the population is growing, we look at the increase in population in the third column. If the population had been growing linearly, all the numbers in the third column would be the same.

Table 1.8 *Population of Burkina Faso (estimated), 2003–2009*

Year	Population (millions)	Change in population (millions)
2003	12.853	
		0.437
2004	13.290	
		0.457
2005	13.747	
		0.478
2006	14.225	
		0.496
2007	14.721	
		0.513
2008	15.234	
		0.523
2009	15.757	

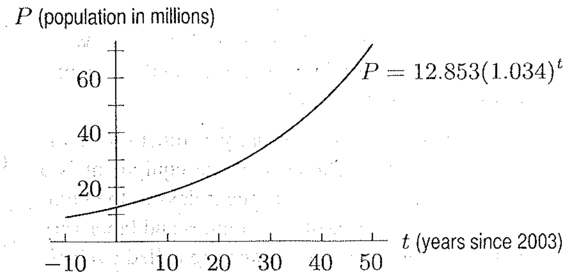


Figure 1.17: Population of Burkina Faso (estimated): Exponential growth

Suppose we divide each year's population by the previous year's population. For example,

$$\frac{\text{Population in 2004}}{\text{Population in 2003}} = \frac{13.290 \text{ million}}{12.853 \text{ million}} = 1.034$$

$$\frac{\text{Population in 2005}}{\text{Population in 2004}} = \frac{13.747 \text{ million}}{13.290 \text{ million}} = 1.034.$$

The fact that both calculations give 1.034 shows the population grew by about 3.4% between 2003 and 2004 *and* between 2004 and 2005. Similar calculations for other years show that the population grew by a factor of about 1.034, or 3.4%, every year. Whenever we have a constant growth factor (here 1.034), we have exponential growth. The population t years after 2003 is given by the *exponential* function

$$P = 12.853(1.034)^t.$$

If we assume that the formula holds for 50 years, the population graph has the shape shown in Figure 1.17. Since the population is growing faster and faster as time goes on, the graph is bending upward; we say it is *concave up*. Even exponential functions which climb slowly at first, such as this one, eventually climb extremely quickly.

> To recognize that a table of t and P values comes from an exponential function, look for ratios of P values that are constant for equally spaced t values.

Concavity

We have used the term concave up[12] to describe the graph in Figure 1.17. In words:

> The graph of a function is **concave up** if it bends upward as we move left to right; it is **concave down** if it bends downward. (See Figure 1.18 for four possible shapes.) A line is neither concave up nor concave down.

[11] dataworldbank.org, accessed January 12, 2011.
[12] In Chapter 2 we consider concavity in more depth.

Figure 1.18: Concavity of a graph

Elimination of a Drug from the Body

Now we look at a quantity which is decreasing exponentially instead of increasing. When a patient is given medication, the drug enters the bloodstream. As the drug passes through the liver and kidneys, it is metabolized and eliminated at a rate that depends on the particular drug. For the antibiotic ampicillin, approximately 40% of the drug is eliminated every hour. A typical dose of ampicillin is 250 mg. Suppose $Q = f(t)$, where Q is the quantity of ampicillin, in mg, in the bloodstream at time t hours since the drug was given. At $t = 0$, we have $Q = 250$. Since every hour the amount remaining is 60% of the previous amount, we have

$$f(0) = 250$$
$$f(1) = 250(0.6)$$
$$f(2) = (250(0.6))(0.6) = 250(0.6)^2,$$

and after t hours,

$$Q = f(t) = 250(0.6)^t.$$

This is an *exponential decay function*. Some values of the function are in Table 1.9; its graph is in Figure 1.19.

Notice the way in which the function in Figure 1.19 is decreasing. Each hour a smaller quantity of the drug is removed than in the previous hour. This is because as time passes, there is less of the drug in the body to be removed. Compare this to the exponential growth in Figure 1.17, where each step upward is larger than the previous one. Notice, however, that both graphs are concave up.

Table 1.9 *Drug elimination*

t (hours)	Q (mg)
0	250
1	150
2	90
3	54
4	32.4
5	19.4

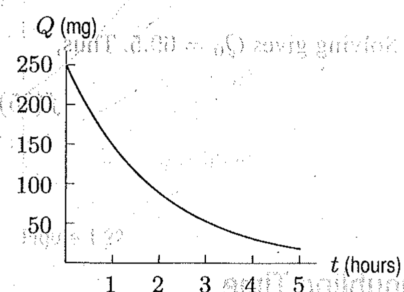


Figure 1.19: Drug elimination: Exponential decay

The General Exponential Function

We say P is an **exponential function** of t with base a if

$$P = P_0 a^t,$$

where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1.

If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.

Provided $a > 0$, the largest possible domain for the exponential function is all real numbers. The reason we do not want $a \leq 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 1^t = P_0$ is then a constant function.

The value of a is closely related to the percent growth (or decay) rate. For example, if $a = 1.03$, then P is growing at 3%; if $a = 0.94$, then P is decaying at 6%.

Example 1 Suppose that $Q = f(t)$ is an exponential function of t. If $f(20) = 88.2$ and $f(23) = 91.4$:

(a) Find the base. (b) Find the growth rate. (c) Evaluate $f(25)$.

Solution (a) Let

$$Q = Q_0 a^t.$$

Substituting $t = 20, Q = 88.2$ and $t = 23, Q = 91.4$ gives two equations for Q_0 and a:

$$88.2 = Q_0 a^{20} \quad \text{and} \quad 91.4 = Q_0 a^{23}.$$

Dividing the two equations enables us to eliminate Q_0:

$$\frac{91.4}{88.2} = \frac{Q_0 a^{23}}{Q_0 a^{20}} = a^3.$$

Solving for the base, a, gives

$$a = \left(\frac{91.4}{88.2}\right)^{1/3} = 1.012.$$

(b) Since $a = 1.012$, the growth rate is $0.012 = 1.2\%$.

(c) We want to evaluate $f(25) = Q_0 a^{25} = Q_0(1.012)^{25}$. First we find Q_0 from the equation

$$88.2 = Q_0(1.012)^{20}.$$

Solving gives $Q_0 = 69.5$. Thus,

$$f(25) = 69.5(1.012)^{25} = 93.6.$$

Half-Life and Doubling Time

Radioactive substances, such as uranium, decay exponentially. A certain percentage of the mass disintegrates in a given unit of time; the time it takes for half the mass to decay is called the *half-life* of the substance.

A well-known radioactive substance is carbon-14, which is used to date organic objects. When a piece of wood or bone was part of a living organism, it accumulated small amounts of radioactive carbon-14. Once the organism dies, it no longer picks up carbon-14. Using the half-life of carbon-14 (about 5730 years), we can estimate the age of the object. We use the following definitions:

> The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.
> The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.

The Family of Exponential Functions

The formula $P = P_0 a^t$ gives a family of exponential functions with positive parameters P_0 (the initial quantity) and a (the base, or growth/decay factor). The base tells us whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.20 and 1.21.) All members of the family $P = P_0 a^t$ are concave up.

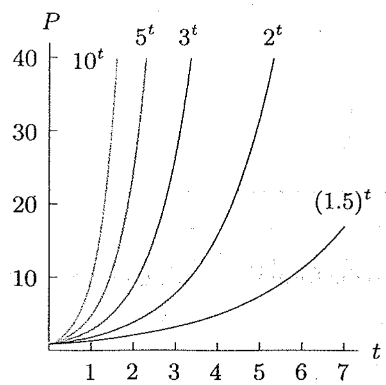


Figure 1.20: Exponential growth: $P = a^t$, for $a > 1$

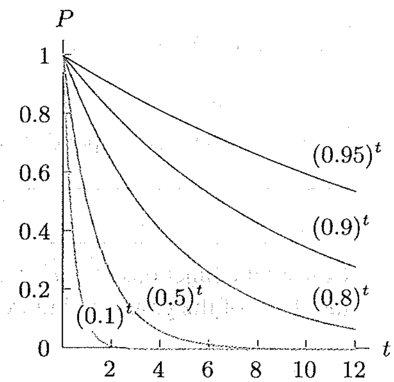


Figure 1.21: Exponential decay: $P = a^t$, for $0 < a < 1$

Example 2 Figure 1.22 is the graph of three exponential functions. What can you say about the values of the six constants, a, b, c, d, p, q?

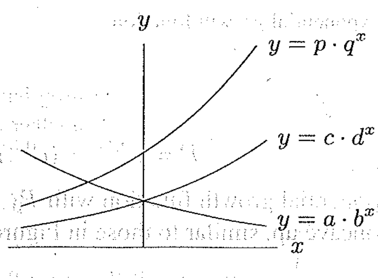


Figure 1.22

Solution All the constants are positive. Since a, c, p represent y-intercepts, we see that $a = c$ because these graphs intersect on the y-axis. In addition, $a = c < p$, since $y = p \cdot q^x$ crosses the y-axis above the other two.

Since $y = a \cdot b^x$ is decreasing, we have $0 < b < 1$. The other functions are increasing, so $1 < d$ and $1 < q$.

Exponential Functions with Base e

The most frequently used base for an exponential function is the famous number $e = 2.71828\ldots$. This base is used so often that you will find an e^x button on most scientific calculators. At first glance, this is all somewhat mysterious. Why is it convenient to use the base $2.71828\ldots$? The full answer to that question must wait until Chapter 3, where we show that many calculus formulas come out neatly when e is used as the base. We often use the following result:

Any **exponential growth** function can be written, for some $a > 1$ and $k > 0$, in the form

$$P = P_0 a^t \quad \text{or} \quad P = P_0 e^{kt}$$

and any **exponential decay** function can be written, for some $0 < a < 1$ and $-k < 0$, as

$$Q = Q_0 a^t \quad \text{or} \quad Q = Q_0 e^{-kt},$$

where P_0 and Q_0 are the initial quantities.

We say that P and Q are growing or decaying at a *continuous*[13] *rate* of k. (For example, $k = 0.02$ corresponds to a continuous rate of 2%.)

Example 3 Convert the functions $P = e^{0.5t}$ and $Q = 5e^{-0.2t}$ into the form $y = y_0 a^t$. Use the results to explain the shape of the graphs in Figures 1.23 and 1.24.

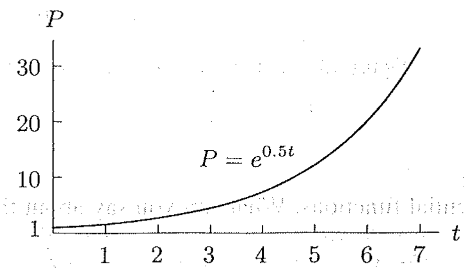


Figure 1.23: An exponential growth function

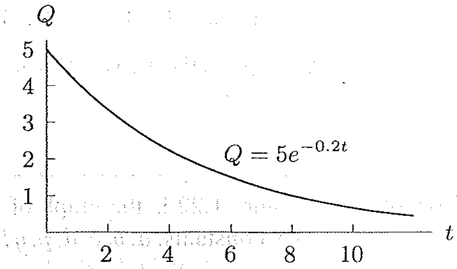


Figure 1.24: An exponential decay function

Solution We have

$$P = e^{0.5t} = (e^{0.5})^t = (1.65)^t.$$

Thus, P is an exponential growth function with $P_0 = 1$ and $a = 1.65$. The function is increasing and its graph is concave up, similar to those in Figure 1.20. Also,

$$Q = 5e^{-0.2t} = 5(e^{-0.2})^t = 5(0.819)^t,$$

so Q is an exponential decay function with $Q_0 = 5$ and $a = 0.819$. The function is decreasing and its graph is concave up, similar to those in Figure 1.21.

Example 4 The quantity, Q, of a drug in a patient's body at time t is represented for positive constants S and k by the function $Q = S(1 - e^{-kt})$. For $t \geq 0$, describe how Q changes with time. What does S represent?

Solution The graph of Q is shown in Figure 1.25. Initially none of the drug is present, but the quantity increases with time. Since the graph is concave down, the quantity increases at a decreasing rate. This is realistic because as the quantity of the drug in the body increases, so does the rate at which the body excretes the drug. Thus, we expect the quantity to level off. Figure 1.25 shows that S is the saturation level. The line $Q = S$ is called a *horizontal asymptote*.

[13]The reason that k is called the continuous rate is explored in detail in Chapter 11.

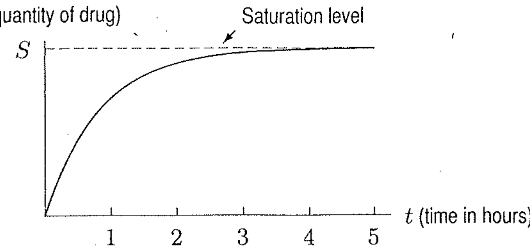


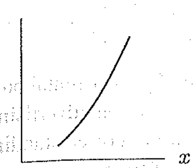
Figure 1.25: Buildup of the quantity of a drug in body

Exercises and Problems for Section 1.2

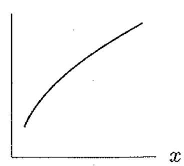
Exercises

In Exercises 1–4, decide whether the graph is concave up, concave down, or neither.

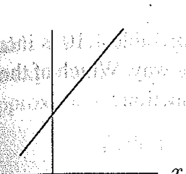
1.



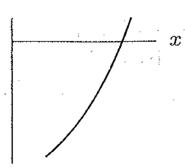
2.



3.



4.



The functions in Exercises 5–8 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

5. $P = 5(1.07)^t$
6. $P = 7.7(0.92)^t$
7. $P = 3.2e^{0.03t}$
8. $P = 15e^{-0.06t}$

Write the functions in Exercises 9–12 in the form $P = P_0a^t$. Which represent exponential growth and which represent exponential decay?

9. $P = 15e^{0.25t}$
10. $P = 2e^{-0.5t}$
11. $P = P_0e^{0.2t}$
12. $P = 7e^{-\pi t}$

In Exercises 13–14, let $f(t) = Q_0a^t = Q_0(1 + r)^t$.
(a) Find the base, a.
(b) Find the percentage growth rate, r.

13. $f(5) = 75.94$ and $f(7) = 170.86$
14. $f(0.02) = 25.02$ and $f(0.05) = 25.06$
15. A town has a population of 1000 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.
(a) The population increases by 50 people a year.
(b) The population increases by 5% a year.

16. An air-freshener starts with 30 grams and evaporates. In each of the following cases, write a formula for the quantity, Q grams, of air-freshener remaining t days after the start and sketch a graph of the function. The decrease is:
(a) 2 grams a day **(b)** 12% a day

17. For which pairs of consecutive points in Figure 1.26 is the function graphed:
(a) Increasing and concave up?
(b) Increasing and concave down?
(c) Decreasing and concave up?
(d) Decreasing and concave down?

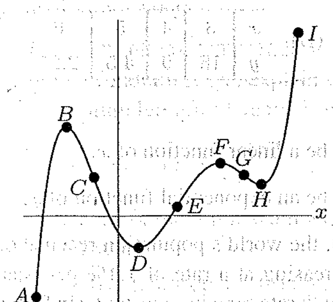


Figure 1.26

18. The table gives the average temperature in Wallingford, Connecticut, for the first 10 days in March.
(a) Over which intervals was the average temperature increasing? Decreasing?
(b) Find a pair of consecutive intervals over which the average temperature was increasing at a decreasing rate. Find another pair of consecutive intervals over which the average temperature was increasing at an increasing rate.

Day	1	2	3	4	5	6	7	8	9	10
°F	42°	42°	34°	25°	22°	34°	38°	40°	49°	49°

Problems

19. **(a)** Which (if any) of the functions in the following table could be linear? Find formulas for those functions.
(b) Which (if any) of these functions could be exponential? Find formulas for those functions.

x	$f(x)$	$g(x)$	$h(x)$
-2	12	16	37
-1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

In Problems 20–21, find all the tables that have the given characteristic.

(A)

x	0	40	80	160
y	2.2	2.2	2.2	2.2

(B)

x	-8	-4	0	8
y	51	62	73	95

(C)

x	-4	-3	4	6
y	18	0	4.5	-2.25

(D)

x	3	4	5	6
y	18	9	4.5	2.25

20. y could be a linear function of x.

21. y could be an exponential function of x.

22. In 2010, the world's population reached 6.91 billion and was increasing at a rate of 1.1% per year. Assume that this growth rate remains constant. (In fact, the growth rate has decreased since 1987.)

(a) Write a formula for the world population (in billions) as a function of the number of years since 2010.
(b) Estimate the population of the world in the year 2020.
(c) Sketch world population as a function of years since 2010. Use the graph to estimate the doubling time of the population of the world.

23. **(a)** A population, P, grows at a continuous rate of 2% a year and starts at 1 million. Write P in the form $P = P_0e^{kt}$, with P_0, k constants.
(b) Plot the population in part (a) against time.

24. A certain region has a population of 10,000,000 and an annual growth rate of 2%. Estimate the doubling time by guessing and checking.

25. A photocopy machine can reduce copies to 80% of their original size. By copying an already reduced copy, further reductions can be made.

(a) If a page is reduced to 80%, what percent enlargement is needed to return it to its original size?
(b) Estimate the number of times in succession that a page must be copied to make the final copy less than 15% of the size of the original.

26. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time goes on.

(a) Graph the total number of people who have tried such a product against time.
(b) What do you know about the concavity of the graph?

27. Sketch reasonable graphs for the following. Pay particular attention to the concavity of the graphs.

(a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.
(b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.

28. Each of the functions g, h, k in Table 1.10 is increasing, but each increases in a different way. Which of the graphs in Figure 1.27 best fits each function?

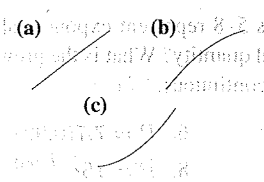


Figure 1.27

Table 1.10

t	$g(t)$	$h(t)$	$k(t)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

29. Each of the functions in Table 1.11 decreases, but each decreases in a different way. Which of the graphs in Figure 1.28 best fits each function?

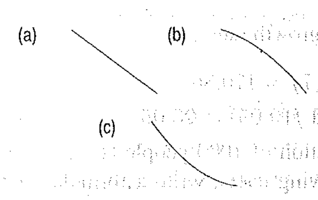


Figure 1.28

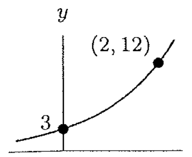
Table 1.11

x	$f(x)$	$g(x)$	$h(x)$
1	100	22.0	9.3
2	90	21.4	9.1
3	81	20.8	8.8
4	73	20.2	8.4
5	66	19.6	7.9
6	60	19.0	7.3

30. One of the main contaminants of a nuclear accident, such as that at Chernobyl, is strontium-90, which decays exponentially at a continuous rate of approximately 2.47% per year. After the Chernobyl disaster, it was suggested that it would be about 100 years before the region would again be safe for human habitation. What percent of the original strontium-90 would still remain then?

Give a possible formula for the functions in Problems 31–34.

31.



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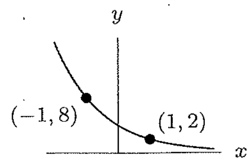

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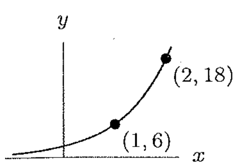

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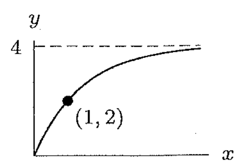


35. Table 1.12 shows some values of a linear function f and an exponential function g. Find exact values (not decimal approximations) for each of the missing entries.

Table 1.12

x	0	1	2	3	4
$f(x)$	10	?	20	?	?
$g(x)$	10	?	20	?	?

36. Match the functions $h(s)$, $f(s)$, and $g(s)$, whose values are in Table 1.13, with the formulas

$$y = a(1.1)^s, \quad y = b(1.05)^s, \quad y = c(1.03)^s,$$

assuming a, b, and c are constants. Note that the function values have been rounded to two decimal places.

Table 1.13

s	$h(s)$	s	$f(s)$	s	$g(s)$
2	1.06	1	2.20	3	3.47
3	1.09	2	2.42	4	3.65
4	1.13	3	2.66	5	3.83
5	1.16	4	2.93	6	4.02
6	1.19	5	3.22	7	4.22

37. **(a)** Estimate graphically the doubling time of the exponentially growing population shown in Figure 1.29. Check that the doubling time is independent of where you start on the graph.

(b) Show algebraically that if $P = P_0a^t$ doubles between time t and time $t + d$, then d is the same number for any t.

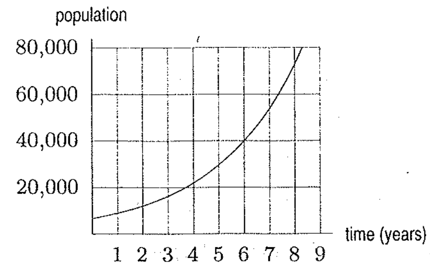


Figure 1.29

38. A deposit of P_0 into a bank account has a doubling time of 50 years. No other deposits or withdrawals are made.

(a) How much money is in the bank account after 50 years? 100 years? 150 years? (Your answer will involve P_0.)

(b) How many times does the amount of money double in t years? Use this to write a formula for P, the amount of money in the account after t years.

39. A 325 mg aspirin has a half-life of H hours in a patient's body.

(a) How long does it take for the quantity of aspirin in the patient's body to be reduced to 162.5 mg? To 81.25 mg? To 40.625 mg? (Note that $162.5 = 325/2$, etc. Your answers will involve H.)

(b) How many times does the quantity of aspirin, A mg, in the body halve in t hours? Use this to give a formula for A after t hours.

40. **(a)** The half-life of radium-226 is 1620 years. If the initial quantity of radium is Q_0, explain why the quantity, Q, of radium left after t years, is given by

$$Q = Q_0\left(\frac{1}{2}\right)^{t/1620}.$$

(b) What percentage of the original amount of radium is left after 500 years?

41. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 2010? [Hint: Write the function in the form $Q = Q_0(1/2)^{t/29}$.]

42. Aircraft require longer takeoff distances, called takeoff rolls, at high altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0° C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

Elevation (ft)	Sea level	1000	2000	3000	4000
Takeoff roll (ft)	670	734	805	882	967

Problems 43–44 concern biodiesel, a fuel derived from renewable resources such as food crops, algae, and animal oils. The table shows the percent growth over the previous year in US biodiesel consumption.[14]

Year	2003	2004	2005	2006	2007	2008	2009
% growth	−12.5	92.9	237	186.6	37.2	−11.7	7.3

43. **(a)** According to the US Department of Energy, the US consumed 91 million gallons of biodiesel in 2005. Approximately how much biodiesel (in millions of gallons) did the US consume in 2006? In 2007?

(b) Graph the points showing the annual US consumption of biodiesel, in millions of gallons of biodiesel, for the years 2005 to 2009. Label the scales on the horizontal and vertical axes.

44. **(a)** True or false: The annual US consumption of biodiesel grew exponentially from 2003 to 2005. Justify your answer without doing any calculations.

(b) According to this data, during what single year(s), if any, did the US consumption of biodiesel at least double?

(c) According to this data, during what single year(s), if any, did the US consumption of biodiesel at least triple?

45. Hydroelectric power is electric power generated by the force of moving water. The table shows the annual percent change in hydroelectric power consumption by the US industrial sector.[15]

Year	2005	2006	2007	2008	2009
% growth over previous yr	−1.9	−10	−45.4	5.1	11

(a) According to the US Department of Energy, the US industrial sector consumed about 29 trillion BTUs of hydroelectric power in 2006. Approximately how much hydroelectric power (in trillion BTUs) did the US consume in 2007? In 2005?

(b) Graph the points showing the annual US consumption of hydroelectric power, in trillion BTUs, for the years 2004 to 2009. Label the scales on the horizontal and vertical axes.

(c) According to this data, when did the largest yearly decrease, in trillion BTUs, in the US consumption of hydroelectric power occur? What was this decrease?

Problems 46–47 concern wind power, which has been used for centuries to propel ships and mill grain. Modern wind power is obtained from windmills which convert wind energy into electricity. Figure 1.30 shows the annual percent growth in US wind power consumption[16] between 2005 and 2009.

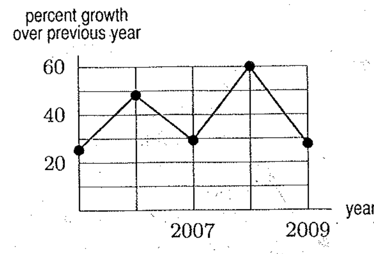


Figure 1.30

46. **(a)** According to the US Department of Energy, the US consumption of wind power was 341 trillion BTUs in 2007. How much wind power did the US consume in 2006? In 2008?

(b) Graph the points showing the annual US consumption of wind power, in trillion BTUs, for the years 2005 to 2009. Label the scales on the horizontal and vertical axes.

(c) Based on this data, in what year did the largest yearly increase, in trillion BTUs, in the US consumption of wind power occur? What was this increase?

47. **(a)** According to Figure 1.30, during what single year(s), if any, did the US consumption of wind power energy increase by at least 40%? Decrease by at least 40%?

(b) Did the US consumption of wind power energy double from 2006 to 2008?

Strengthen Your Understanding

In Problems 48–49, explain what is wrong with the statement.

48. The function $y = e^{-0.25x}$ is decreasing and its graph is concave down.

49. The function $y = 2x$ is increasing, and its graph is concave up.

In Problems 50–52, give an example of:

50. A formula representing the statement "q decreases at a constant percent rate, and $q = 2.2$ when $t = 0$."

51. A function that is increasing at a constant percent rate and that has the same vertical intercept as $f(x) = 0.3x + 2$.

52. A function with a horizontal asymptote at $y = -5$ and range $y > -5$.

[14] http://www.eia.doe.gov/aer/renew.html. Accessed February 2011.
[15] From http://www.eia.doe.gov/aer/renew.html. Accessed February 2011.
[16] Yearly values have been joined with segments to highlight trends in the data. Actual values in between years should not be inferred from the segments. From http://www.eia.doe.gov/aer/renew.html. Accessed February 2011.

Are the statements in Problems 53–59 true or false? Give an explanation for your answer.

53. The function $y = 2 + 3e^{-t}$ has a y-intercept of $y = 3$.
54. The function $y = 5 - 3e^{-4t}$ has a horizontal asymptote of $y = 5$.
55. If $y = f(x)$ is an exponential function and if increasing x by 1 increases y by a factor of 5, then increasing x by 2 increases y by a factor of 10.
56. If $y = Ab^x$ and increasing x by 1 increases y by a factor of 3, then increasing x by 2 increases y by a factor of 9.
57. An exponential function can be decreasing.
58. If a and b are positive constants, $b \neq 1$, then $y = a + ab^x$ has a horizontal asymptote.
59. The function $y = 20/(1 + 2e^{-kt})$ with $k > 0$, has a horizontal asymptote at $y = 20$.

1.3 NEW FUNCTIONS FROM OLD

Shifts and Stretches

The graph of a constant multiple of a given function is easy to visualize: each y-value is stretched or shrunk by that multiple. For example, consider the function $f(x)$ and its multiples $y = 3f(x)$ and $y = -2f(x)$. Their graphs are shown in Figure 1.31. The factor 3 in the function $y = 3f(x)$ stretches each $f(x)$ value by multiplying it by 3; the factor -2 in the function $y = -2f(x)$ stretches $f(x)$ by multiplying by 2 and reflects it about the x-axis. You can think of the multiples of a given function as a family of functions.

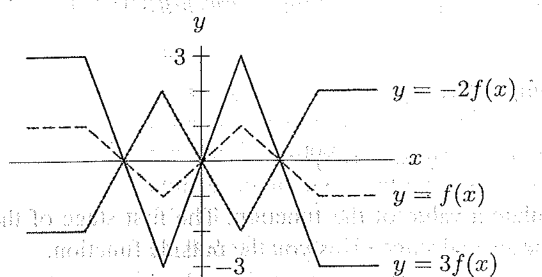


Figure 1.31: Multiples of the function $f(x)$

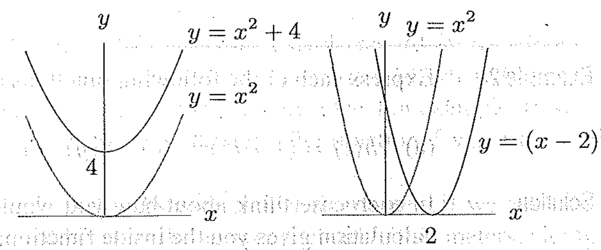


Figure 1.32: Graphs of $y = x^2$ with $y = x^2 + 4$ and $y = (x - 2)^2$

It is also easy to create families of functions by shifting graphs. For example, $y - 4 = x^2$ is the same as $y = x^2 + 4$, which is the graph of $y = x^2$ shifted up by 4. Similarly, $y = (x - 2)^2$ is the graph of $y = x^2$ shifted right by 2. (See Figure 1.32.)

- Multiplying a function by a constant, c, stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$). A negative sign (if $c < 0$) reflects the graph about the x-axis, in addition to shrinking or stretching.
- Replacing y by $(y - k)$ moves a graph up by k (down if k is negative).
- Replacing x by $(x - h)$ moves a graph to the right by h (to the left if h is negative).

Composite Functions

If oil is spilled from a tanker, the area of the oil slick grows with time. Suppose that the oil slick is always a perfect circle. Then the area, A, of the oil slick is a function of its radius, r:

$$A = f(r) = \pi r^2.$$

The radius is also a function of time, because the radius increases as more oil spills. Thus, the area, being a function of the radius, is also a function of time. If, for example, the radius is given by

$$r = g(t) = 1 + t,$$

then the area is given as a function of time by substitution:

$$A = \pi r^2 = \pi(1+t)^2.$$

We are thinking of A as a *composite function* or a "function of a function," which is written

$$A = \underbrace{f(g(t))}_{\substack{\text{Composite function;} \\ f \text{ is outside function,} \\ g \text{ is inside function}}} = \pi(g(t))^2 = \pi(1+t)^2.$$

To calculate A using the formula $\pi(1+t)^2$, the first step is to find $1+t$, and the second step is to square and multiply by π. The first step corresponds to the inside function $g(t) = 1+t$, and the second step corresponds to the outside function $f(r) = \pi r^2$.

Example 1 If $f(x) = x^2$ and $g(x) = x - 2$, find each of the following:
(a) $f(g(3))$ (b) $g(f(3))$ (c) $f(g(x))$ (d) $g(f(x))$

Solution (a) Since $g(3) = 1$, we have $f(g(3)) = f(1) = 1$.
(b) Since $f(3) = 9$, we have $g(f(3)) = g(9) = 7$. Notice that $f(g(3)) \neq g(f(3))$.
(c) $f(g(x)) = f(x-2) = (x-2)^2$.
(d) $g(f(x)) = g(x^2) = x^2 - 2$. Again, notice that $f(g(x)) \neq g(f(x))$.
Notice that the horizontal shift in Figure 1.32 can be thought of as a composition $f(g(x)) = (x-2)^2$.

Example 2 Express each of the following functions as a composition:

(a) $h(t) = (1+t^3)^{27}$ (b) $k(y) = e^{-y^2}$ (c) $l(y) = -(e^y)^2$

Solution In each case think about how you would calculate a value of the function. The first stage of the calculation gives you the inside function, and the second stage gives you the outside function.

(a) For $(1+t^3)^{27}$, the first stage is cubing and adding 1, so an inside function is $g(t) = 1+t^3$. The second stage is taking the 27th power, so an outside function is $f(y) = y^{27}$. Then

$$f(g(t)) = f(1+t^3) = (1+t^3)^{27}.$$

In fact, there are lots of different answers: $g(t) = t^3$ and $f(y) = (1+y)^{27}$ is another possibility.

(b) To calculate e^{-y^2} we square y, take its negative, and then take e to that power. So if $g(y) = -y^2$ and $f(z) = e^z$, then we have

$$f(g(y)) = e^{-y^2}.$$

(c) To calculate $-(e^y)^2$, we find e^y, square it, and take the negative. Using the same definitions of f and g as in part (b), the composition is

$$g(f(y)) = -(e^y)^2.$$

Since parts (b) and (c) give different answers, we see the order in which functions are composed is important.

Odd and Even Functions: Symmetry

There is a certain symmetry apparent in the graphs of $f(x) = x^2$ and $g(x) = x^3$ in Figure 1.33. For each point (x, x^2) on the graph of f, the point $(-x, x^2)$ is also on the graph; for each point (x, x^3) on the graph of g, the point $(-x, -x^3)$ is also on the graph. The graph of $f(x) = x^2$ is symmetric about the y-axis, whereas the graph of $g(x) = x^3$ is symmetric about the origin. The graph of any polynomial involving only even powers of x has symmetry about the y-axis, while polynomials with only odd powers of x are symmetric about the origin. Consequently, any functions with these symmetry properties are called *even* and *odd*, respectively.

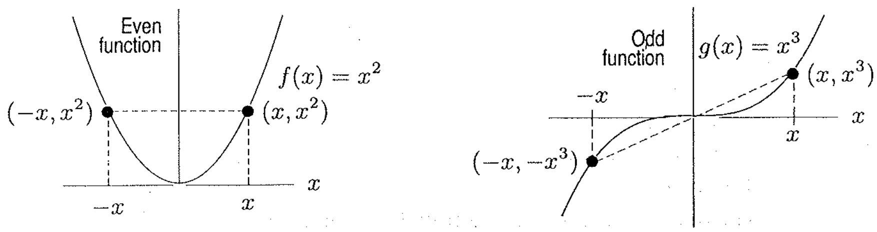


Figure 1.33: Symmetry of even and odd functions

> For any function f,
> f is an **even** function if $f(-x) = f(x)$ for all x.
> f is an **odd** function if $f(-x) = -f(x)$ for all x.

For example, $g(x) = e^{x^2}$ is even and $h(x) = x^{1/3}$ is odd. However, many functions do not have any symmetry and are neither even nor odd.

Inverse Functions

On August 26, 2005, the runner Kenenisa Bekele[17] of Ethiopia set a world record for the 10,000-meter race. His times, in seconds, at 2000-meter intervals are recorded in Table 1.14, where $t = f(d)$ is the number of seconds Bekele took to complete the first d meters of the race. For example, Bekele ran the first 4000 meters in 629.98 seconds, so $f(4000) = 629.98$. The function f was useful to athletes planning to compete with Bekele.

Let us now change our point of view and ask for distances rather than times. If we ask how far Bekele ran during the first 629.98 seconds of his race, the answer is clearly 4000 meters. Going backward in this way from numbers of seconds to numbers of meters gives f^{-1}, the *inverse function*[18] of f. We write $f^{-1}(629.98) = 4000$. Thus, $f^{-1}(t)$ is the number of meters that Bekele ran during the first t seconds of his race. See Table 1.15, which contains values of f^{-1}.

The independent variable for f is the dependent variable for f^{-1}, and vice versa. The domains and ranges of f and f^{-1} are also interchanged. The domain of f is all distances d such that $0 \leq d \leq 10000$, which is the range of f^{-1}. The range of f is all times t, such that $0 \leq t \leq 1577.53$, which is the domain of f^{-1}.

Table 1.14 *Bekele's running time*

d (meters)	$t = f(d)$ (seconds)
0	0.00
2000	315.63
4000	629.98
6000	944.66
8000	1264.63
10000	1577.53

Table 1.15 *Distance run by Bekele*

t (seconds)	$d = f^{-1}(t)$ (meters)
0.00	0
315.63	2000
629.98	4000
944.66	6000
1264.63	8000
1577.53	10000

Which Functions Have Inverses?

If a function has an inverse, we say it is *invertible*. Let's look at a function which is not invertible. Consider the flight of the Mercury spacecraft *Freedom 7*, which carried Alan Shepard, Jr. into space

[17] kenenisabekelle.com/, accessed January 11, 2011.
[18] The notation f^{-1} represents the inverse function, which is not the same as the reciprocal, $1/f$.

in May 1961. Shepard was the first American to journey into space. After launch, his spacecraft rose to an altitude of 116 miles, and then came down into the sea. The function $f(t)$ giving the altitude in miles t minutes after lift-off does not have an inverse. To see why not, try to decide on a value for $f^{-1}(100)$, which should be the time when the altitude of the spacecraft was 100 miles. However, there are two such times, one when the spacecraft was ascending and one when it was descending. (See Figure 1.34.)

The reason the altitude function does not have an inverse is that the altitude has the same value for two different times. The reason the Bekele time function did have an inverse is that each running time, t, corresponds to a unique distance, d.

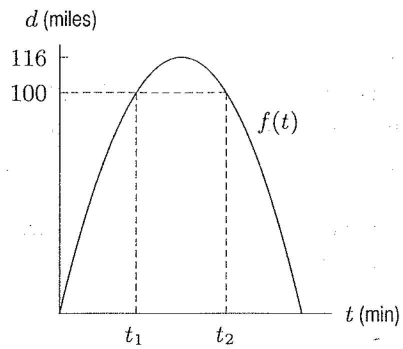


Figure 1.34: Two times, t_1 and t_2, at which altitude of spacecraft is 100 miles

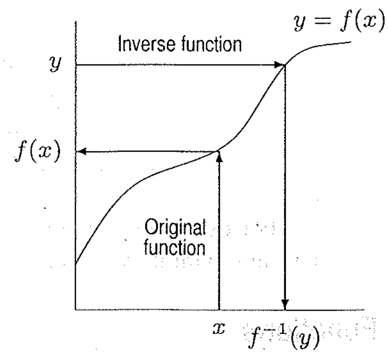


Figure 1.35: A function which has an inverse

Figure 1.35 suggests when an inverse exists. The original function, f, takes us from an x-value to a y-value, as shown in Figure 1.35. Since having an inverse means there is a function going from a y-value to an x-value, the crucial question is whether we can get back. In other words, does each y-value correspond to a unique x-value? If so, there's an inverse; if not, there is not. This principle may be stated geometrically, as follows:

> A function has an inverse if (and only if) its graph intersects any horizontal line at most once.

For example, the function $f(x) = x^2$ does not have an inverse because many horizontal lines intersect the parabola twice.

Definition of an Inverse Function

If the function f is invertible, its inverse is defined as follows:

$$f^{-1}(y) = x \quad \text{means} \quad y = f(x).$$

Formulas for Inverse Functions

If a function is defined by a formula, it is sometimes possible to find a formula for the inverse function. In Section 1.1, we looked at the snow tree cricket, whose chirp rate, C, in chirps per minute, is approximated at the temperature, T, in degrees Fahrenheit, by the formula

$$C = f(T) = 4T - 160.$$

So far we have used this formula to predict the chirp rate from the temperature. But it is also possible to use this formula backward to calculate the temperature from the chirp rate.

Example 3 Find the formula for the function giving temperature in terms of the number of cricket chirps per minute; that is, find the inverse function f^{-1} such that

$$T = f^{-1}(C).$$

Solution Since C is an increasing function, f is invertible. We know $C = 4T - 160$. We solve for T, giving

$$T = \frac{C}{4} + 40,$$

so

$$f^{-1}(C) = \frac{C}{4} + 40.$$

Graphs of Inverse Functions

The function $f(x) = x^3$ is increasing everywhere and so has an inverse. To find the inverse, we solve

$$y = x^3$$

for x, giving

$$x = y^{1/3}.$$

The inverse function is

$$f^{-1}(y) = y^{1/3}$$

or, if we want to call the independent variable x,

$$f^{-1}(x) = x^{1/3}.$$

The graphs of $y = x^3$ and $y = x^{1/3}$ are shown in Figure 1.36. Notice that these graphs are the reflections of one another about the line $y = x$. For example, $(8, 2)$ is on the graph of $y = x^{1/3}$ because $2 = 8^{1/3}$, and $(2, 8)$ is on the graph of $y = x^3$ because $8 = 2^3$. The points $(8, 2)$ and $(2, 8)$ are reflections of one another about the line $y = x$.

In general, we have the following result.

> If the x- and y-axes have the same scales, the graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

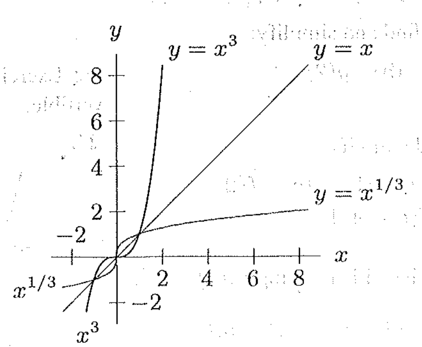


Figure 1.36: Graphs of inverse functions, $y = x^3$ and $y = x^{1/3}$, are reflections about the line $y = x$

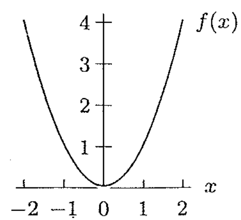
Exercises and Problems for Section 1.3

Exercises

For the functions f in Exercises 1–3, graph:

(a) $f(x+2)$ **(b)** $f(x-1)$ **(c)** $f(x)-4$
(d) $f(x+1)+3$ **(e)** $3f(x)$ **(f)** $-f(x)+1$

1.



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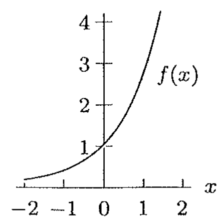

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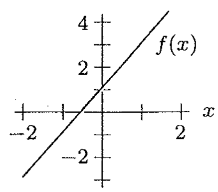


In Exercises 4–7, use Figure 1.37 to graph the functions.

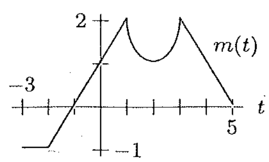


Figure 1.37

4. $n(t)=m(t)+2$
5. $p(t)=m(t-1)$
6. $k(t)=m(t+1.5)$
7. $w(t)=m(t-0.5)-2.5$

For the functions f and g in Exercises 8–11, find

(a) $f(g(1))$ **(b)** $g(f(1))$ **(c)** $f(g(x))$
(d) $g(f(x))$ **(e)** $f(t)g(t)$

8. $f(x)=x^2, g(x)=x+1$
9. $f(x)=\sqrt{x+4}, g(x)=x^2$
10. $f(x)=e^x, g(x)=x^2$
11. $f(x)=1/x, g(x)=3x+4$
12. For $g(x)=x^2+2x+3$, find and simplify:
(a) $g(2+h)$ **(b)** $g(2)$
(c) $g(2+h)-g(2)$
13. If $f(x)=x^2+1$, find and simplify:
(a) $f(t+1)$ **(b)** $f(t^2+1)$ **(c)** $f(2)$
(d) $2f(t)$ **(e)** $(f(t))^2+1$

Simplify the quantities in Exercises 14–17 using $m(z)=z^2$.

14. $m(z+1)-m(z)$
15. $m(z+h)-m(z)$
16. $m(z)-m(z-h)$
17. $m(z+h)-m(z-h)$

18. Let p be the price of an item and q be the number of items sold at that price, where $q=f(p)$. What do the following quantities mean in terms of prices and quantities sold?
(a) $f(25)$ **(b)** $f^{-1}(30)$

19. Let $C=f(A)$ be the cost, in dollars, of building a store of area A square feet. In terms of cost and square feet, what do the following quantities represent?
(a) $f(10{,}000)$ **(b)** $f^{-1}(20{,}000)$

20. Let $f(x)$ be the temperature (°F) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(75)$ in practical terms?

21. **(a)** Write an equation for a graph obtained by vertically stretching the graph of $y=x^2$ by a factor of 2, followed by a vertical upward shift of 1 unit. Sketch it.
(b) What is the equation if the order of the transformations (stretching and shifting) in part (a) is interchanged?
(c) Are the two graphs the same? Explain the effect of reversing the order of transformations.

22. Use Figure 1.38 to graph each of the following. Label any intercepts or asymptotes that can be determined.
(a) $y=f(x)+3$ **(b)** $y=2f(x)$
(c) $y=f(x+4)$ **(d)** $y=4-f(x)$

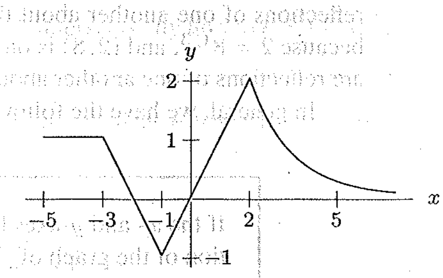
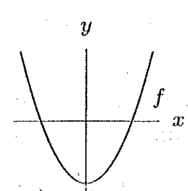


Figure 1.38

For Exercises 23–24, decide if the function $y=f(x)$ is invertible.

23.



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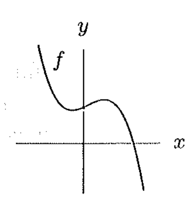


For Exercises 25–27, use a graph of the function to decide whether or not it is invertible.

25. $f(x) = x^2 + 3x + 2$ **26.** $f(x) = x^3 - 5x + 10$

27. $f(x) = x^3 + 5x + 10$

Are the functions in Exercises 28–35 even, odd, or neither?

28. $f(x) = x^6 + x^3 + 1$ **29.** $f(x) = x^3 + x^2 + x$

30. $f(x) = x^4 - x^2 + 3$ **31.** $f(x) = x^3 + 1$

32. $f(x) = 2x$ **33.** $f(x) = e^{x^2-1}$

34. $f(x) = x(x^2 - 1)$ **35.** $f(x) = e^x - x$

Problems

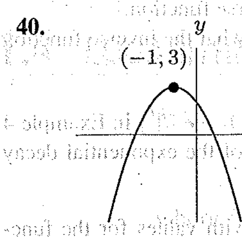
For Problems 36–39, determine functions f and g such that $h(x) = f(g(x))$. [Note: There is more than one correct answer. Do not choose $f(x) = x$ or $g(x) = x$.]

36. $h(x) = (x+1)^3$ **37.** $h(x) = x^3 + 1$

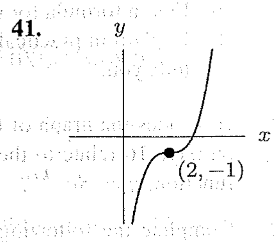
38. $h(x) = \sqrt{x^2 + 4}$ **39.** $h(x) = e^{2x}$

Find possible formulas for the graphs in Problems 40–41 using shifts of x^2 or x^3.

40.



41.



42. **(a)** Use Figure 1.39 to estimate $f^{-1}(2)$.
(b) Sketch a graph of f^{-1} on the same axes.

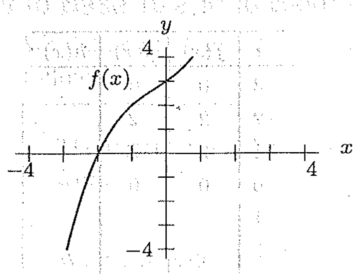


Figure 1.39

43. Write a table of values for f^{-1}, where f is as given below. The domain of f is the integers from 1 to 7. State the domain of f^{-1}.

x	1	2	3	4	5	6	7
$f(x)$	3	−7	19	4	178	2	1

For Problems 44–47, decide if the function f is invertible.

44. $f(d)$ is the total number of gallons of fuel an airplane has used by the end of d minutes of a particular flight.

45. $f(t)$ is the number of customers in Macy's department store at t minutes past noon on December 18, 2008.

46. $f(n)$ is the number of students in your calculus class whose birthday is on the n^{th} day of the year.

47. $f(w)$ is the cost of mailing a letter weighing w grams.

In Problems 48–51 the functions $r = f(t)$ and $V = g(r)$ give the radius and the volume of a commercial hot air balloon being inflated for testing. The variable t is in minutes, r is in feet, and V is in cubic feet. The inflation begins at $t = 0$. In each case, give a mathematical expression that represents the given statement.

48. The volume of the balloon t minutes after inflation began.

49. The volume of the balloon if its radius were twice as big.

50. The time that has elapsed when the radius of the balloon is 30 feet.

51. The time that has elapsed when the volume of the balloon is 10,000 cubic feet.

In Problems 52–55, use Figure 1.40 to estimate the function value or explain why it cannot be done.

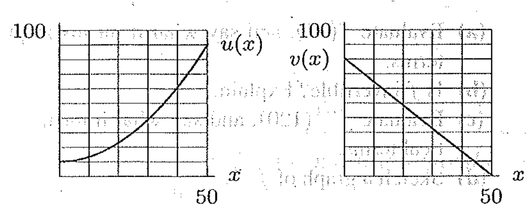


Figure 1.40

52. $u(v(10))$ **53.** $u(v(40))$

54. $v(u(10))$ **55.** $v(u(40))$

56. Figure 1.41 shows $f(t)$, the number (in millions) of motor vehicles registered[19] in the world in the year t.

(a) Is f invertible? Explain.
(b) What is the meaning of $f^{-1}(400)$ in practical terms? Evaluate $f^{-1}(400)$.
(c) Sketch the graph of f^{-1}.

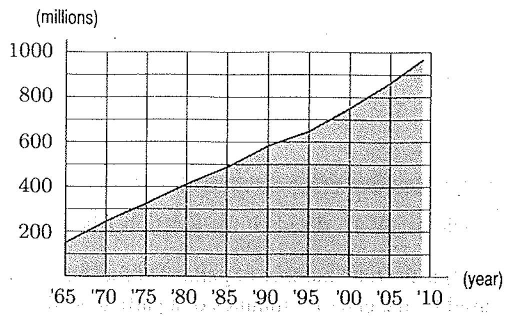


Figure 1.41

For Problems 57–62, use the graphs in Figure 1.42.

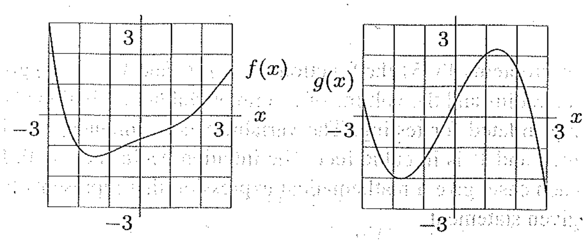


Figure 1.42

57. Estimate $f(g(1))$.

58. Estimate $g(f(2))$.

59. Estimate $f(f(1))$.

60. Graph $f(g(x))$.

61. Graph $g(f(x))$.

62. Graph $f(f(x))$.

63. Figure 1.43 is a graph of the function $f(t)$. Here $f(t)$ is the depth in meters below the Atlantic Ocean floor where t million-year-old rock can be found.[20]

(a) Evaluate $f(15)$, and say what it means in practical terms.
(b) Is f invertible? Explain.
(c) Evaluate $f^{-1}(120)$, and say what it means in practical terms.
(d) Sketch a graph of f^{-1}.

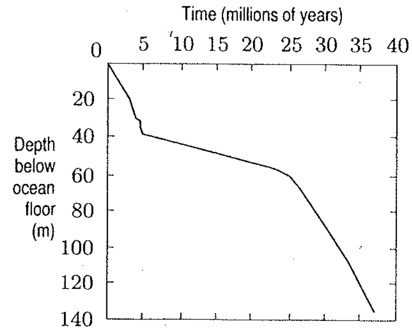


Figure 1.43

64. A tree of height y meters has, on average, B branches, where $B = y - 1$. Each branch has, on average, n leaves, where $n = 2B^2 - B$. Find the average number of leaves of a tree as a function of height.

65. A spherical balloon is growing with radius $r = 3t + 1$, in centimeters, for time t in seconds. Find the volume of the balloon at 3 seconds.

66. The cost of producing q articles is given by the function $C = f(q) = 100 + 2q$.

(a) Find a formula for the inverse function.
(b) Explain in practical terms what the inverse function tells you.

67. How does the graph of $Q = S(1 - e^{-kt})$ in Example 4 on page 16 relate to the graph of the exponential decay function, $y = Se^{-kt}$?

68. Complete the following table with values for the functions f, g, and h, given that:

(a) f is an even function.
(b) g is an odd function.
(c) h is the composition $h(x) = g(f(x))$.

x	$f(x)$	$g(x)$	$h(x)$
−3	0	0	
−2	2	2	
−1	2	2	
0	0	0	
1			
2			
3			

[19] www.earth-policy.org, accessed June 5, 2011. In 2000, about 30% of the registered vehicles were in the US.
[20] Data of Dr. Murlene Clark based on core samples drilled by the research ship *Glomar Challenger*, taken from *Initial Reports of the Deep Sea Drilling Project.*

Strengthen Your Understanding

In Problems 69–71, explain what is wrong with the statement.

69. The graph of $f(x) = -(x+1)^3$ is the graph of $g(x) = -x^3$ shifted right by 1 unit.

70. $f(x) = 3x+5$ and $g(x) = -3x-5$ are inverse functions of each other.

71. The inverse of $f(x) = x$ is $f^{-1}(x) = 1/x$.

In Problems 72–75, give an example of:

72. An invertible function whose graph contains the point $(0, 3)$.

73. An even function whose graph does not contain the point $(0, 0)$.

74. An increasing function $f(x)$ whose values are greater than those of its inverse function $f^{-1}(x)$ for $x > 0$.

75. Two functions $f(x)$ and $g(x)$ such that moving the graph of f to the left 2 units gives the graph of g and moving the graph of f up 3 also gives the graph of g.

Are the statements in Problems 76–83 true or false? Give an explanation for your answer.

76. The graph of $f(x) = 100(10^x)$ is a horizontal shift of the graph of $g(x) = 10^x$.

77. If f is an increasing function, then f^{-1} is an increasing function.

78. If a function is even, then it does not have an inverse.

79. If a function is odd, then it does not have an inverse.

80. The function $f(x) = e^{-x^2}$ is decreasing for all x.

81. If $g(x)$ is an even function then $f(g(x))$ is even for every function $f(x)$.

82. If $f(x)$ is an even function then $f(g(x))$ is even for every function $g(x)$.

83. There is a function which is both even and odd.

Suppose f is an increasing function and g is a decreasing function. In Problems 84–87, give an example for f and g for which the statement is true, or say why such an example is impossible.

84. $f(x) + g(x)$ is decreasing for all x.

85. $f(x) - g(x)$ is decreasing for all x.

86. $f(x)g(x)$ is decreasing for all x.

87. $f(g(x))$ is increasing for all x.

1.4 LOGARITHMIC FUNCTIONS

In Section 1.2, we approximated the population of Burkina Faso (in millions) by the function

$$P = f(t) = 12.853(1.034)^t,$$

where t is the number of years since 2003. Now suppose that instead of calculating the population at time t, we ask when the population will reach 20 million. We want to find the value of t for which

$$20 = f(t) = 12.853(1.034)^t.$$

We use logarithms to solve for a variable in an exponent.

Logarithms to Base 10 and to Base e

We define the *logarithm* function, $\log_{10} x$, to be the inverse of the exponential function, 10^x, as follows:

> The **logarithm** to base 10 of x, written $\log_{10} x$, is the power of 10 we need to get x. In other words,
>
> $$\log_{10} x = c \quad \text{means} \quad 10^c = x.$$
>
> We often write $\log x$ in place of $\log_{10} x$.

The other frequently used base is e. The logarithm to base e is called the *natural logarithm* of x, written $\ln x$ and defined to be the inverse function of e^x, as follows:

> The **natural logarithm** of x, written $\ln x$, is the power of e needed to get x. In other words,
>
> $$\ln x = c \quad \text{means} \quad e^c = x.$$

Values of $\log x$ are in Table 1.16. Because no power of 10 gives 0, $\log 0$ is undefined. The graph of $y = \log x$ is shown in Figure 1.44. The domain of $y = \log x$ is positive real numbers; the range is all real numbers. In contrast, the inverse function $y = 10^x$ has domain all real numbers and range all positive real numbers. The graph of $y = \log x$ has a vertical asymptote at $x = 0$, whereas $y = 10^x$ has a horizontal asymptote at $y = 0$.

One big difference between $y = 10^x$ and $y = \log x$ is that the exponential function grows extremely quickly whereas the log function grows extremely slowly. However, $\log x$ does go to infinity, albeit slowly, as x increases. Since $y = \log x$ and $y = 10^x$ are inverse functions, the graphs of the two functions are reflections of one another about the line $y = x$, provided the scales along the x- and y-axes are equal.

Table 1.16 *Values for* $\log x$ *and* 10^x

x	$\log x$
0	undefined
1	0
2	0.3
3	0.5
4	0.6
$\vdots$	$\vdots$
10	1

x	10^x
0	1
1	10
2	100
3	10^3
4	10^4
$\vdots$	$\vdots$
10	10^{10}

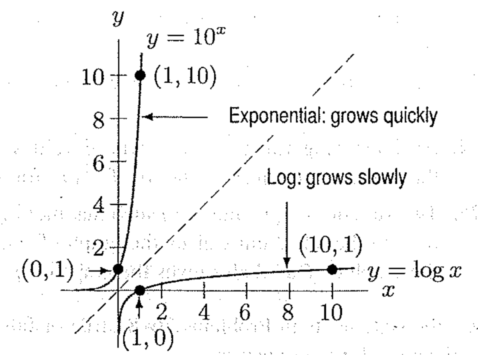


Figure 1.44: Graphs of $\log x$ and 10^x

The graph of $y = \ln x$ in Figure 1.45 has roughly the same shape as the graph of $y = \log x$. The x-intercept is $x = 1$, since $\ln 1 = 0$.The graph of $y = \ln x$ also climbs very slowly as x increases. Both graphs, $y = \log x$ and $y = \ln x$, have *vertical asymptotes* at $x = 0$.

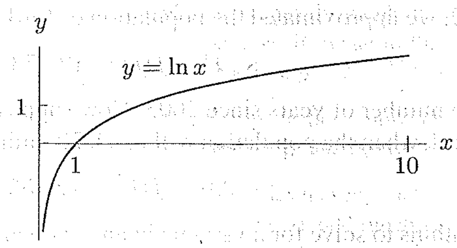


Figure 1.45: Graph of the natural logarithm

The following properties of logarithms may be deduced from the properties of exponents:

Properties of Logarithms

Note that $\log x$ and $\ln x$ are not defined when x is negative or 0.

1. $\log(AB) = \log A + \log B$	1. $\ln(AB) = \ln A + \ln B$
2. $\log\left(\frac{A}{B}\right) = \log A - \log B$	2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3. $\log(A^p) = p \log A$	3. $\ln(A^p) = p \ln A$
4. $\log(10^x) = x$	4. $\ln e^x = x$
5. $10^{\log x} = x$	5. $e^{\ln x} = x$

In addition, $\log 1 = 0$ because $10^0 = 1$, and $\ln 1 = 0$ because $e^0 = 1$.

Solving Equations Using Logarithms

Logs are frequently useful when we have to solve for unknown exponents, as in the next examples.

Example 1 Find t such that $2^t = 7$.

Solution First, notice that we expect t to be between 2 and 3 (because $2^2 = 4$ and $2^3 = 8$). To calculate t, we take logs to base 10 of both sides. (Natural logs could also be used.)

$$\log(2^t) = \log 7.$$

Then use the third property of logs, which says $\log(2^t) = t\log 2$, and get:

$$t\log 2 = \log 7.$$

Using a calculator to find the logs gives

$$t = \frac{\log 7}{\log 2} \approx 2.81.$$

Example 2 Find when the population of Burkina Faso reaches 20 million by solving $20 = 12.853(1.034)^t$.

Solution Dividing both sides of the equation by 12.853, we get

$$\frac{20}{12.853} = (1.034)^t.$$

Now take logs of both sides:

$$\log\left(\frac{20}{12.853}\right) = \log(1.034^t).$$

Using the fact that $\log(A^t) = t\log A$, we get

$$\log\left(\frac{20}{12.853}\right) = t\log(1.034).$$

Solving this equation using a calculator to find the logs, we get

$$t = \frac{\log(20/12.853)}{\log(1.034)} = 13.22 \text{ years}$$

which is between $t = 13$ and $t = 14$. This value of t corresponds to the year 2016.

Example 3 Traffic pollution is harmful to school-age children. The concentration of carbon monoxide, CO, in the air near a busy road is a function of distance from the road. The concentration decays exponentially at a continuous rate of 3.3% per meter.[21] At what distance from the road is the concentration of CO half what it is on the road?

Solution If C_0 is the concentration of CO on the road, then the concentration x meters from the road is

$$C = C_0 e^{-0.033x}.$$

We want to find the value of x making $C = C_0/2$, that is,

$$C_0 e^{-0.033x} = \frac{C_0}{2}.$$

Dividing by C_0 and then taking natural logs yields

$$\ln\left(e^{-0.033x}\right) = -0.033x = \ln\left(\frac{1}{2}\right) = -0.6931,$$

so

$$x = 21 \text{ meters}.$$

At 21 meters from the road the concentration of CO in the air is half the concentration on the road.

[21] Rickwood, P. and Knight, D. (2009). "The health impacts of local traffic pollution on primary school age children." *State of Australian Cities 2009 Conference Proceedings*.

In Example 3 the decay rate was given. However, in many situations where we expect to find exponential growth or decay, the rate is not given. To find it, we must know the quantity at two different times and then solve for the growth or decay rate, as in the next example.

Example 4 The population of Mexico was 99.9 million in 2000 and 113.4 million in 2010.[22] Assuming it increases exponentially, find a formula for the population of Mexico as a function of time.

Solution If we measure the population, P, in millions and time, t, in years since 2000, we can say

$$P = P_0 e^{kt} = 99.9e^{kt},$$

where $P_0 = 99.9$ is the initial value of P. We find k by using the fact that $P = 113.4$ when $t = 10$, so

$$113.4 = 99.9e^{k \cdot 10}.$$

To find k, we divide both sides by 99.9, giving

$$\frac{113.4}{99.9} = 1.135 = e^{10k}.$$

Now take natural logs of both sides:

$$\ln(1.135) = \ln(e^{10k}).$$

Using a calculator and the fact that $\ln(e^{10k}) = 10k$, this becomes

$$0.127 = 10k.$$

So

$$k = 0.0127,$$

and therefore

$$P = 99.9e^{0.0127t}.$$

Since $k = 0.0127 = 1.27\%$, the population of Mexico was growing at a continuous rate of 1.27% per year.

In Example 4 we chose to use e for the base of the exponential function representing Mexico's population, making clear that the continuous growth rate was 1.27%. If we had wanted to emphasize the annual growth rate, we could have expressed the exponential function in the form $P = P_0 a^t$.

Example 5 Give a formula for the inverse of the following function (that is, solve for t in terms of P):

$$P = f(t) = 12.853(1.034)^t.$$

Solution We want a formula expressing t as a function of P. Take logs:

$$\log P = \log(12.853(1.034)^t).$$

Since $\log(AB) = \log A + \log B$, we have

$$\log P = \log 12.853 + \log((1.034)^t).$$

Now use $\log(A^t) = t \log A$:

$$\log P = \log 12.853 + t \log 1.034.$$

Solve for t in two steps, using a calculator at the final stage:

$$t \log 1.034 = \log P - \log 12.853$$

$$t = \frac{\log P}{\log 1.034} - \frac{\log 12.853}{\log 1.034} = 68.868 \log P - 76.375.$$

[22] http://data.worldbank.org/country/mexico. Accessed January 14, 2012.

Thus,

$$f^{-1}(P) = 68.868 \log P - 76.375.$$

Note that

$$f^{-1}(20) = 68.868(\log 20) - 76.375 = 13.22,$$

which agrees with the result of Example 2.

Exercises and Problems for Section 1.4

Exercises

Simplify the expressions in Exercises 1–6 completely.

1. $e^{\ln(1/2)}$

2. $10^{\log(AB)}$

3. $5e^{\ln(A^2)}$

4. $\ln(e^{2AB})$

5. $\ln(1/e) + \ln(AB)$

6. $2\ln\left(e^A\right) + 3\ln B^e$

For Exercises 7–18, solve for x using logs.

7. $3^x = 11$

8. $17^x = 2$

9. $20 = 50(1.04)^x$

10. $4 \cdot 3^x = 7 \cdot 5^x$

11. $7 = 5e^{0.2x}$

12. $2^x = e^{x+1}$

13. $50 = 600e^{-0.4x}$

14. $2e^{3x} = 4e^{5x}$

15. $7^{x+2} = e^{17x}$

16. $10^{x+3} = 5e^{7-x}$

17. $2x - 1 = e^{\ln x^2}$

18. $4e^{2x-3} - 5 = e$

For Exercises 19–24, solve for t. Assume a and b are positive constants and k is nonzero.

19. $a = b^t$

20. $P = P_0\, a^t$

21. $Q = Q_0\, a^{nt}$

22. $P_0\, a^t = Q_0\, b^t$

23. $a = be^t$

24. $P = P_0\, e^{kt}$

In Exercises 25–28, put the functions in the form $P = P_0 e^{kt}$.

25. $P = 15(1.5)^t$

26. $P = 10(1.7)^t$

27. $P = 174(0.9)^t$

28. $P = 4(0.55)^t$

Find the inverse function in Exercises 29–31.

29. $p(t) = (1.04)^t$

30. $f(t) = 50e^{0.1t}$

31. $f(t) = 1 + \ln t$

Problems

32. The population of a region is growing exponentially. There were 40,000,000 people in 2000 ($t = 0$) and 48,000,000 in 2010. Find an expression for the population at any time t, in years. What population would you predict for the year 2020? What is the doubling time?

33. One hundred kilograms of a radioactive substance decay to 40 kg in 10 years. How much remains after 20 years?

34. A culture of bacteria originally numbers 500. After 2 hours there are 1500 bacteria in the culture. Assuming exponential growth, how many are there after 6 hours?

35. The population of the US was 281.4 million in 2000 and 308.7 million in 2010.[23] Assuming exponential growth,

(a) In what year is the population expected to go over 350 million?

(b) What population is predicted for the 2020 census?

36. The concentration of the car exhaust fume nitrous oxide, NO_2, in the air near a busy road is a function of distance from the road. The concentration decays exponentially at a continuous rate of 2.54% per meter.[24] At what distance from the road is the concentration of NO_2 half what it is on the road?

37. For children and adults with diseases such as asthma, the number of respiratory deaths per year increases by 0.33% when pollution particles increase by a microgram per cubic meter of air.[25]

(a) Write a formula for the number of respiratory deaths per year as a function of quantity of pollution in the air. (Let Q_0 be the number of deaths per year with no pollution.)

(b) What quantity of air pollution results in twice as many respiratory deaths per year as there would be without pollution?

[23] http://2010.census.gov/2010census/. Accessed April 17, 2011.

[24] Rickwood, P. and Knight, D. (2009). "The health impacts of local traffic pollution on primary school age children." *State of Australian Cities 2009 Conference Proceedings.*

[25] Brook, R. D., Franklin, B., Cascio, W., Hong, Y., Howard, G., Lipsett, M., Luepker, R., Mittleman, M., Samet, J., and Smith, S. C. (2004). "Air pollution and cardiovascular disease." *Circulation,* 109(21):2655267.

38. The number of alternative fuel vehicles[26] running on E85, fuel that is up to 85% plant-derived ethanol, increased exponentially in the US between 2003 and 2008.

(a) Use this information to complete the missing table values.
(b) How many E85-powered vehicles were there in the US in 2003?
(c) By what percent did the number of E85-powered vehicles grow from 2004 to 2008?

Year	2004	2005	2006	2007	2008
Number of E85 vehicles	211,800	?	?	?	450,327

39. At time t hours after taking the cough suppressant hydrocodone bitartrate, the amount, A, in mg, remaining in the body is given by $A = 10(0.82)^t$.

(a) What was the initial amount taken?
(b) What percent of the drug leaves the body each hour?
(c) How much of the drug is left in the body 6 hours after the dose is administered?
(d) How long is it until only 1 mg of the drug remains in the body?

40. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.
(b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.
(c) Use logarithms to find the half-life of caffeine.

41. The exponential function $y(x) = Ce^{\alpha x}$ satisfies the conditions $y(0) = 2$ and $y(1) = 1$. Find the constants C and α. What is $y(2)$?

42. Without a calculator or computer, match the functions e^x, $\ln x$, x^2, and $x^{1/2}$ to their graphs in Figure 1.46.

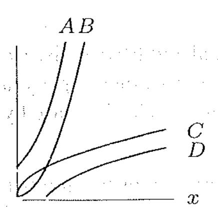


Figure 1.46

43. With time, t, in years since the start of 1980, textbook prices have increased at 6.7% per year while inflation has been 3.3% per year.[27] Assume both rates are continuous growth rates.

(a) Find a formula for $B(t)$, the price of a textbook in year t if it cost $\$B_0$ in 1980.
(b) Find a formula for $P(t)$, the price of an item in year t if it cost $\$P_0$ in 1980 and its price rose according to inflation.
(c) A textbook cost \$50 in 1980. When is its price predicted to be double the price that would have resulted from inflation alone?

44. In November 2010, a "tiger summit" was held in St. Petersburg, Russia.[28] In 1900, there were 100,000 wild tigers worldwide; in 2010 the number was 3200.

(a) Assuming the tiger population has decreased exponentially, find a formula for $f(t)$, the number of wild tigers t years since 1900.
(b) Between 2000 and 2010, the number of wild tigers decreased by 40%. Is this percentage larger or smaller than the decrease in the tiger population predicted by your answer to part (a)?

45. In 2011, the populations of China and India were approximately 1.34 and 1.19 billion people[29], respectively. However, due to central control the annual population growth rate of China was 0.4% while the population of India was growing by 1.37% each year. If these growth rates remain constant, when will the population of India exceed that of China?

46. The third-quarter revenue of Apple® went from \$3.68 billion[30] in 2005 to \$15.68 billion[31] in 2010. Find an exponential function to model the revenue as a function of years since 2005. What is the continuous percent growth rate, per year, of sales?

47. The world population was 6.9 billion at the end of 2010 and is predicted to reach 9 billion by the end of 2050.[32]

(a) Assuming the population is growing exponentially, what is the continuous growth rate per year?
(b) The United Nations celebrated the "Day of 5 Billion" on July 11, 1987, and the "Day of 6 Billion" on October 12, 1999. Using the growth rate in part (a), when is the "Day of 7 Billion" predicted to be?

[26] http://www.eia.doe.gov/aer/renew.html
[27] Data from "Textbooks headed for ash heap of history", http://educationtechnews.com, Vol 5, 2010.
[28] "Tigers would be extinct in Russia if unprotected," Yahoo! News, Nov. 21, 2010.
[29] http://www.indexmundi.com/. Accessed April 17, 2011.
[30] http://www.apple.com/pr/library/2005/oct/11results.html. Accessed April 27, 2011.
[31] http://www.apple.com/pr/library/2010/01/25results.html. Accessed April 27, 2011.
[32] "Reviewing the Bidding on the Climate Files", in About Dot Earth, *New York Times*, Nov. 19, 2010.

48. In the early 1920s, Germany had tremendously high inflation, called hyperinflation. Photographs of the time show people going to the store with wheelbarrows full of money. If a loaf of bread cost 1/4 marks in 1919 and 2,400,000 marks in 1922, what was the average yearly inflation rate between 1919 and 1922?

49. Different isotopes (versions) of the same element can have very different half-lives. With t in years, the decay of plutonium-240 is described by the formula

$$Q = Q_0 e^{-0.00011t},$$

whereas the decay of plutonium-242 is described by

$$Q = Q_0 e^{-0.0000018t}.$$

Find the half-lives of plutonium-240 and plutonium-242.

50. The size of an exponentially growing bacteria colony doubles in 5 hours. How long will it take for the number of bacteria to triple?

51. Air pressure, P, decreases exponentially with height, h, above sea level. If P_0 is the air pressure at sea level and h is in meters, then

$$P = P_0 e^{-0.00012h}.$$

(a) At the top of Mount McKinley, height 6194 meters (about 20,320 feet), what is the air pressure, as a percent of the pressure at sea level?
(b) The maximum cruising altitude of an ordinary commercial jet is around 12,000 meters (about 39,000 feet). At that height, what is the air pressure, as a percent of the sea level value?

52. Find the equation of the line l in Figure 1.47.

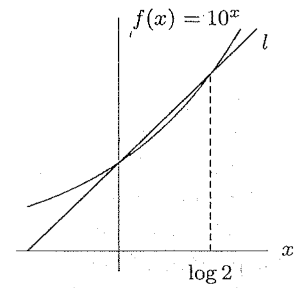


Figure 1.47

53. In 2010, there were about 246 million vehicles (cars and trucks) and about 308.7 million people in the US.[33] The number of vehicles grew 15.5% over the previous decade, while the population has been growing at 9.7% per decade. If the growth rates remain constant, when will there be, on average, one vehicle per person?

54. A picture supposedly painted by Vermeer (1632–1675) contains 99.5% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.

55. Is there a difference between $\ln[\ln(x)]$ and $\ln^2(x)$? [Note: $\ln^2(x)$ is another way of writing $(\ln x)^2$.]

56. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on

(a) The y-intercept? (b) The x-intercept?

57. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on the vertical asymptote?

58. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on

(a) The y-intercept? (b) The x-intercept?

59. If $f(x) = a\ln(x + 2)$, what is the effect of increasing a on the vertical asymptote?

60. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on the vertical asymptote?

Strengthen Your Understanding

In Problems 61–62, explain what is wrong with the statement.

61. The function $-\log|x|$ is odd.

62. For all $x > 0$, the value of $\ln(100x)$ is 100 times larger than $\ln x$.

In Problems 63–64, give an example of:

63. A function $f(x)$ such that $\ln(f(x))$ is only defined for $x < 0$.

64. A function with a vertical asymptote at $x = 3$ and defined only for $x > 3$.

Are the statements in Problems 65–68 true or false? Give an explanation for your answer.

65. The graph of $f(x) = \ln x$ is concave down.

66. The graph of $g(x) = \log(x - 1)$ crosses the x-axis at $x = 1$.

67. The inverse function of $y = \log x$ is $y = 1/\log x$.

68. If a and b are positive constants, then $y = \ln(ax + b)$ has no vertical asymptote.

[33] http://www.autoblog.com/2010/01/04/report-number-of-cars-in-the-u-s-dropped-by-four-million-in-20/ and http://2010.census.gov/news/releases/operations/cb10-cn93.html. Accessed February 2012.

1.5 TRIGONOMETRIC FUNCTIONS

Trigonometry originated as part of the study of triangles. The name *tri-gon-o-metry* means the measurement of three-cornered figures, and the first definitions of the trigonometric functions were in terms of triangles. However, the trigonometric functions can also be defined using the unit circle, a definition that makes them periodic, or repeating. Many naturally occurring processes are also periodic. The water level in a tidal basin, the blood pressure in a heart, an alternating current, and the position of the air molecules transmitting a musical note all fluctuate regularly. Such phenomena can be represented by trigonometric functions.

Radians

There are two commonly used ways to represent the input of the trigonometric functions: radians and degrees. The formulas of calculus, as you will see, are neater in radians than in degrees.

> An angle of 1 **radian** is defined to be the angle at the center of a unit circle which cuts off an arc of length 1, measured counterclockwise. (See Figure 1.48(a).) A unit circle has radius 1.

An angle of 2 radians cuts off an arc of length 2 on a unit circle. A negative angle, such as $-1/2$ radians, cuts off an arc of length $1/2$, but measured clockwise. (See Figure 1.48(b).)

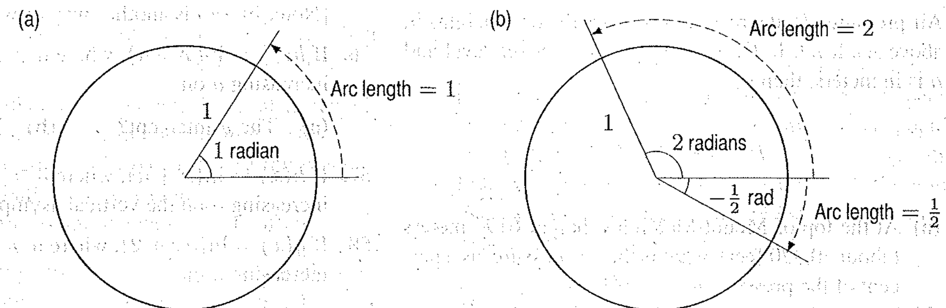


Figure 1.48: Radians defined using unit circle

It is useful to think of angles as rotations, since then we can make sense of angles larger than 360°; for example, an angle of 720° represents two complete rotations counterclockwise. Since one full rotation of 360° cuts off an arc of length 2π, the circumference of the unit circle, it follows that

$$360^\circ = 2\pi \text{ radians}, \quad \text{so} \quad 180^\circ = \pi \text{ radians}.$$

In other words, 1 radian $= 180^\circ/\pi$, so one radian is about 60°. The word radians is often dropped, so if an angle or rotation is referred to without units, it is understood to be in radians.

Radians are useful for computing the length of an arc in any circle. If the circle has radius r and the arc cuts off an angle θ, as in Figure 1.49, then we have the following relation:

$$\text{Arc length} = s = r\theta.$$

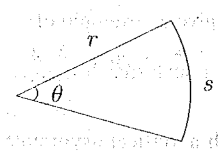


Figure 1.49: Arc length of a sector of a circle

The Sine and Cosine Functions

The two basic trigonometric functions—the sine and cosine—are defined using a unit circle. In Figure 1.50, an angle of t radians is measured counterclockwise around the circle from the point $(1, 0)$. If P has coordinates (x, y), we define

$$\cos t = x \quad \text{and} \quad \sin t = y.$$

We assume that the angles are *always* in radians unless specified otherwise.

Since the equation of the unit circle is $x^2 + y^2 = 1$, writing $\cos^2 t$ for $(\cos t)^2$, we have the identity

$$\cos^2 t + \sin^2 t = 1.$$

As t increases and P moves around the circle, the values of $\sin t$ and $\cos t$ oscillate between 1 and -1, and eventually repeat as P moves through points where it has been before. If t is negative, the angle is measured clockwise around the circle.

Amplitude, Period, and Phase

The graphs of sine and cosine are shown in Figure 1.51. Notice that sine is an odd function, and cosine is even. The maximum and minimum values of sine and cosine are $+1$ and -1, because those are the maximum and minimum values of y and x on the unit circle. After the point P has moved around the complete circle once, the values of $\cos t$ and $\sin t$ start to repeat; we say the functions are *periodic.*

For any periodic function of time, the
- **Amplitude** is half the distance between the maximum and minimum values (if it exists).
- **Period** is the smallest time needed for the function to execute one complete cycle.

The amplitude of $\cos t$ and $\sin t$ is 1, and the period is 2π. Why 2π? Because that's the value of t when the point P has gone exactly once around the circle. (Remember that $360° = 2\pi$ radians.)

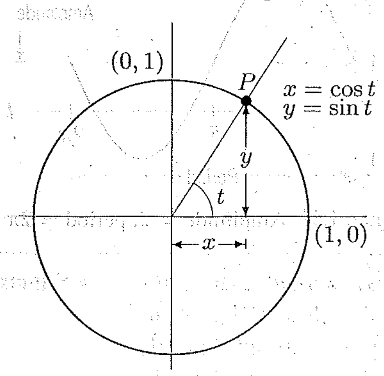


Figure 1.50: The definitions of $\sin t$ and $\cos t$

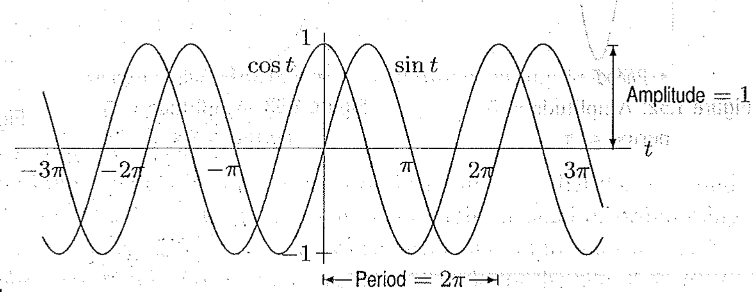


Figure 1.51: Graphs of $\cos t$ and $\sin t$

In Figure 1.51, we see that the sine and cosine graphs are exactly the same shape, only shifted horizontally. Since the cosine graph is the sine graph shifted $\pi/2$ to the left,

$$\cos t = \sin(t + \pi/2).$$

Equivalently, the sine graph is the cosine graph shifted $\pi/2$ to the right, so

$$\sin t = \cos(t - \pi/2).$$

We say that the *phase difference* or *phase shift* between $\sin t$ and $\cos t$ is $\pi/2$.

Functions whose graphs are the shape of a sine or cosine curve are called *sinusoidal* functions.

> To describe arbitrary amplitudes and periods of sinusoidal functions, we use functions of the form
> $$f(t) = A\sin(Bt) \qquad \text{and} \qquad g(t) = A\cos(Bt),$$
> where $|A|$ is the amplitude and $2\pi/|B|$ is the period.
>
> The graph of a sinusoidal function is shifted horizontally by a distance $|h|$ when t is replaced by $t - h$ or $t + h$.
>
> Functions of the form $f(t) = A\sin(Bt) + C$ and $g(t) = A\cos(Bt) + C$ have graphs which are shifted vertically by C and oscillate about this value.

Example 1 Find and show on a graph the amplitude and period of the functions

(a) $y = 5\sin(2t)$ (b) $y = -5\sin\left(\frac{t}{2}\right)$ (c) $y = 1 + 2\sin t$

Solution

(a) From Figure 1.52, you can see that the amplitude of $y = 5\sin(2t)$ is 5 because the factor of 5 stretches the oscillations up to 5 and down to -5. The period of $y = \sin(2t)$ is π, because when t changes from 0 to π, the quantity $2t$ changes from 0 to 2π, so the sine function goes through one complete oscillation.

(b) Figure 1.53 shows that the amplitude of $y = -5\sin(t/2)$ is again 5, because the negative sign reflects the oscillations in the t-axis, but does not change how far up or down they go. The period of $y = -5\sin(t/2)$ is 4π because when t changes from 0 to 4π, the quantity $t/2$ changes from 0 to 2π, so the sine function goes through one complete oscillation.

(c) The 1 shifts the graph $y = 2\sin t$ up by 1. Since $y = 2\sin t$ has an amplitude of 2 and a period of 2π, the graph of $y = 1 + 2\sin t$ goes up to 3 and down to -1, and has a period of 2π. (See Figure 1.54.) Thus, $y = 1 + 2\sin t$ also has amplitude 2.

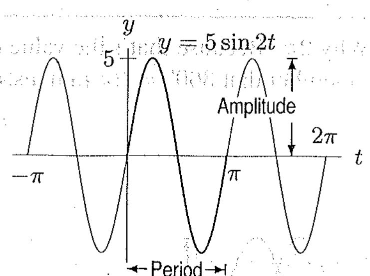


Figure 1.52: Amplitude = 5, period = π

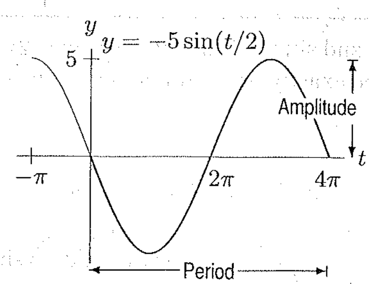


Figure 1.53: Amplitude = 5, period = 4π

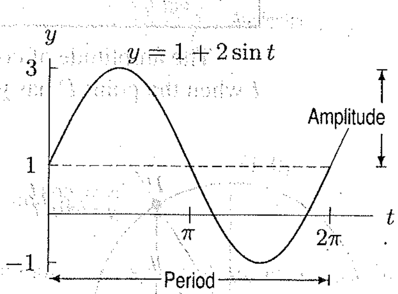
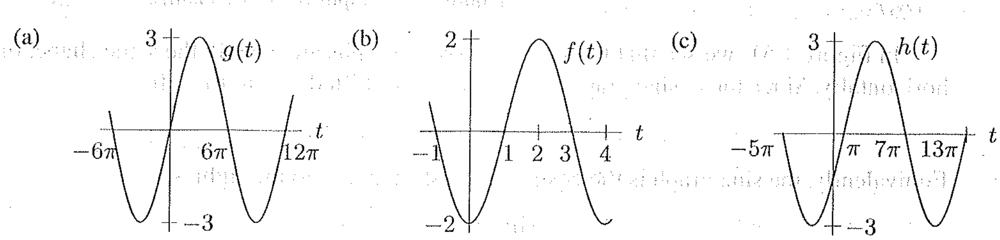


Figure 1.54: Amplitude = 2, period = 2π

Example 2 Find possible formulas for the following sinusoidal functions.



Solution

(a) This function looks like a sine function with amplitude 3, so $g(t) = 3\sin(Bt)$. Since the function executes one full oscillation between $t = 0$ and $t = 12\pi$, when t changes by 12π, the quantity Bt changes by 2π. This means $B \cdot 12\pi = 2\pi$, so $B = 1/6$. Therefore, $g(t) = 3\sin(t/6)$ has the graph shown.

(b) This function looks like an upside-down cosine function with amplitude 2, so $f(t) = -2\cos(Bt)$. The function completes one oscillation between $t = 0$ and $t = 4$. Thus, when t changes by 4, the quantity Bt changes by 2π, so $B \cdot 4 = 2\pi$, or $B = \pi/2$. Therefore, $f(t) = -2\cos(\pi t/2)$ has the graph shown.

(c) This function looks like the function $g(t)$ in part (a), but shifted a distance of π to the right. Since $g(t) = 3\sin(t/6)$, we replace t by $(t - \pi)$ to obtain $h(t) = 3\sin[(t - \pi)/6]$.

Example 3 On July 1, 2007, high tide in Boston was at midnight. The water level at high tide was 9.9 feet; later, at low tide, it was 0.1 feet. Assuming the next high tide is at exactly 12 noon and that the height of the water is given by a sine or cosine curve, find a formula for the water level in Boston as a function of time.

Solution Let y be the water level in feet, and let t be the time measured in hours from midnight. The oscillations have amplitude 4.9 feet $(= (9.9 - 0.1)/2)$ and period 12, so $12B = 2\pi$ and $B = \pi/6$. Since the water is highest at midnight, when $t = 0$, the oscillations are best represented by a cosine function. (See Figure 1.55.) We can say

$$\text{Height above average} = 4.9\cos\left(\frac{\pi}{6}t\right).$$

Since the average water level was 5 feet $(= (9.9 + 0.1)/2)$, we shift the cosine up by adding 5:

$$y = 5 + 4.9\cos\left(\frac{\pi}{6}t\right).$$

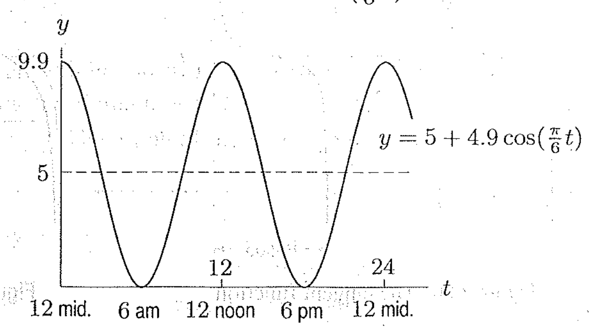


Figure 1.55: Function approximating the tide in Boston on July 1, 2007

Example 4 Of course, there's something wrong with the assumption in Example 3 that the next high tide is at noon. If so, the high tide would always be at noon or midnight, instead of progressing slowly through the day, as in fact it does. The interval between successive high tides actually averages about 12 hours 24 minutes. Using this, give a more accurate formula for the height of the water as a function of time.

Solution The period is 12 hours 24 minutes = 12.4 hours, so $B = 2\pi/12.4$, giving

$$y = 5 + 4.9\cos\left(\frac{2\pi}{12.4}t\right) = 5 + 4.9\cos(0.507t).$$

Example 5 Use the information from Example 4 to write a formula for the water level in Boston on a day when the high tide is at 2 pm.

Solution When the high tide is at midnight,

$$y = 5 + 4.9\cos(0.507t).$$

Since 2 pm is 14 hours after midnight, we replace t by $(t - 14)$. Therefore, on a day when the high tide is at 2 pm,

$$y = 5 + 4.9\cos(0.507(t - 14)).$$

The Tangent Function

If t is any number with $\cos t \neq 0$, we define the tangent function as follows

$$\tan t = \frac{\sin t}{\cos t}.$$

Figure 1.50 on page 37 shows the geometrical meaning of the tangent function: $\tan t$ is the slope of the line through the origin $(0, 0)$ and the point $P = (\cos t, \sin t)$ on the unit circle.

The tangent function is undefined wherever $\cos t = 0$, namely, at $t = \pm\pi/2, \pm 3\pi/2, \ldots$, and it has a vertical asymptote at each of these points. The function $\tan t$ is positive where $\sin t$ and $\cos t$ have the same sign. The graph of the tangent is shown in Figure 1.56.

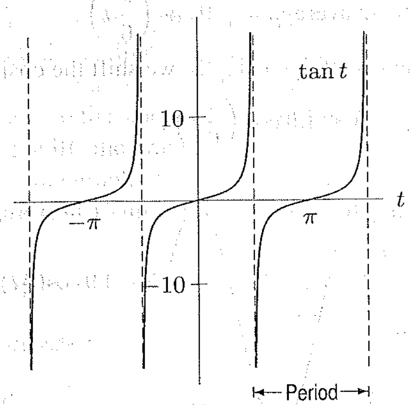


Figure 1.56: The tangent function

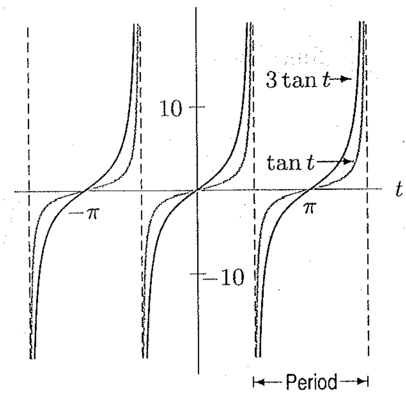


Figure 1.57: Multiple of tangent

The tangent function has period π, because it repeats every π units. Does it make sense to talk about the amplitude of the tangent function? Not if we're thinking of the amplitude as a measure of the size of the oscillation, because the tangent becomes infinitely large near each vertical asymptote. We can still multiply the tangent by a constant, but that constant no longer represents an amplitude. (See Figure 1.57.)

The Inverse Trigonometric Functions

On occasion, you may need to find a number with a given sine. For example, you might want to find x such that

$$\sin x = 0$$

or such that

$$\sin x = 0.3.$$

The first of these equations has solutions $x = 0, \pm\pi, \pm 2\pi, \ldots$. The second equation also has infinitely many solutions. Using a calculator and a graph, we get

$$x \approx 0.305, 2.84, 0.305 \pm 2\pi, 2.84 \pm 2\pi, \ldots.$$

For each equation, we pick out the solution between $-\pi/2$ and $\pi/2$ as the preferred solution. For example, the preferred solution to $\sin x = 0$ is $x = 0$, and the preferred solution to $\sin x = 0.3$ is $x = 0.305$. We define the inverse sine, written "arcsin" or "$\sin^{-1}$," as the function which gives the preferred solution.

> For $-1 \le y \le 1$,
> $$\arcsin y = x$$
> means $$\sin x = y \quad \text{with} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Thus the arcsine is the inverse function to the piece of the sine function having domain $[-\pi/2, \pi/2]$. (See Table 1.17 and Figure 1.58.) On a calculator, the arcsine function[34] is usually denoted by $\boxed{\sin^{-1}}$.

Table 1.17 *Values of* $\sin x$ *and* $\sin^{-1} x$

x	$\sin x$	x	$\sin^{-1} x$
$-\frac{\pi}{2}$	-1.000	-1.000	$-\frac{\pi}{2}$
-1.0	-0.841	-0.841	-1.0
-0.5	-0.479	-0.479	-0.5
0.0	0.000	0.000	0.0
0.5	0.479	0.479	0.5
1.0	0.841	0.841	1.0
$\frac{\pi}{2}$	1.000	1.000	$\frac{\pi}{2}$

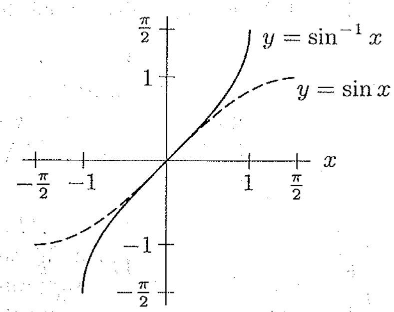


Figure 1.58: The arcsine function

The inverse tangent, written "arctan" or "$\tan^{-1}$," is the inverse function for the piece of the tangent function having the domain $-\pi/2 < x < \pi/2$. On a calculator, the inverse tangent is usually denoted by $\boxed{\tan^{-1}}$. The graph of the arctangent is shown in Figure 1.60.

> For any y,
> $$\arctan y = x$$
> means $$\tan x = y \quad \text{with} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The inverse cosine function, written "arccos" or "$\cos^{-1}$," is discussed in Problem 55. The range of the arccosine function is $0 \le x \le \pi$.

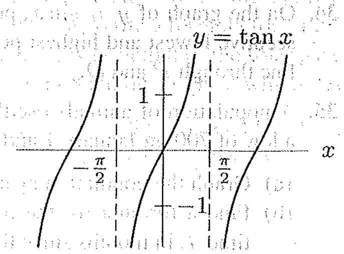


Figure 1.59: The tangent function

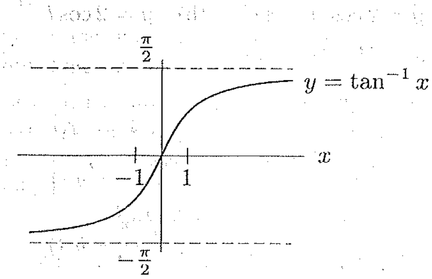


Figure 1.60: The arctangent function

[34]Note that $\sin^{-1} x = \arcsin x$ is not the same as $(\sin x)^{-1} = 1/\sin x$.

Exercises and Problems for Section 1.5

Exercises

For Exercises 1–9, draw the angle using a ray through the origin, and determine whether the sine, cosine, and tangent of that angle are positive, negative, zero, or undefined.

1. $\frac{3\pi}{2}$ **2.** 2π **3.** $\frac{\pi}{4}$

4. 3π **5.** $\frac{\pi}{6}$ **6.** $\frac{4\pi}{3}$

7. $\frac{-4\pi}{3}$ **8.** 4 **9.** -1

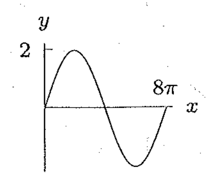
Find the period and amplitude in Exercises 10–13.

10. $y = 7\sin(3t)$ **11.** $z = 3\cos(u/4) + 5$

12. $w = 8 - 4\sin(2x + \pi)$ **13.** $r = 0.1\sin(\pi t) + 2$

For Exercises 14–23, find a possible formula for each graph.

14.



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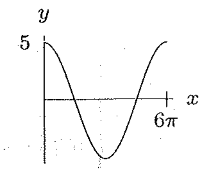

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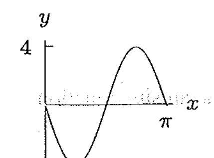

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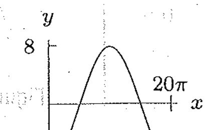

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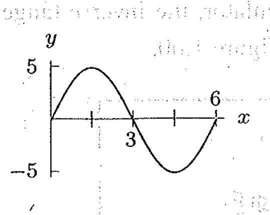

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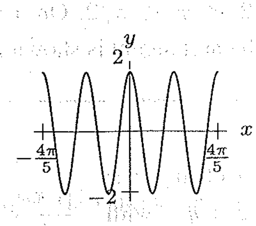

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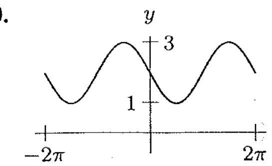

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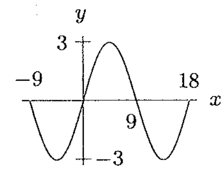

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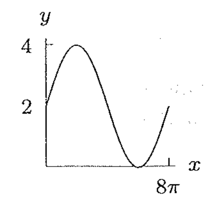

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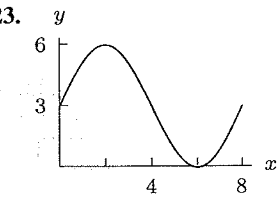

In Exercises 24–26, calculate the quantity without using the the trigonometric functions on your calculator. You are given that $\sin(\pi/12) = 0.259$ and $\cos(\pi/5) = 0.809$. You may want to draw a picture showing the angles involved and check your answer on a calculator.

24. $\cos\left(-\frac{\pi}{5}\right)$ **25.** $\sin\frac{\pi}{5}$ **26.** $\cos\frac{\pi}{12}$

In Exercises 27–31, find a solution to the equation if possible. Give the answer in exact form and in decimal form.

27. $2 = 5\sin(3x)$ **28.** $1 = 8\cos(2x + 1) - 3$

29. $8 = 4\tan(5x)$ **30.** $1 = 8\tan(2x + 1) - 3$

31. $8 = 4\sin(5x)$

Problems

32. Without a calculator or computer, match the formulas with the graphs in Figure 1.61.

(a) $y = 2\cos(t - \pi/2)$ **(b)** $y = 2\cos t$

(c) $y = 2\cos(t + \pi/2)$

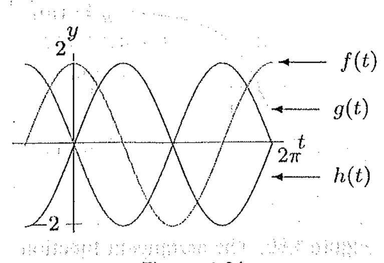

Figure 1.61

33. What is the difference between $\sin x^2$, $\sin^2 x$, and $\sin(\sin x)$? Express each of the three as a composition. (Note: $\sin^2 x$ is another way of writing $(\sin x)^2$.)

34. On the graph of $y = \sin x$, points P and Q are at consecutive lowest and highest points. Find the slope of the line through P and Q.

35. A population of animals oscillates sinusoidally between a low of 700 on January 1 and a high of 900 on July 1.

(a) Graph the population against time.

(b) Find a formula for the population as a function of time, t, in months since the start of the year.

36. The desert temperature, H, oscillates daily between 40°F at 5 am and 80°F at 5 pm. Write a possible formula for H in terms of t, measured in hours from 5 am.

37. **(a)** Match the functions $\omega = f(t), \omega = g(t), \omega = h(t), \omega = k(t)$, whose values are in the table, with the functions with formulas:

(i) $\omega = 1.5 + \sin t$ (ii) $\omega = 0.5 + \sin t$
(iii) $\omega = -0.5 + \sin t$ (iv) $\omega = -1.5 + \sin t$

(b) Based on the table, what is the relationship between the values of $g(t)$ and $k(t)$? Explain this relationship using the formulas you chose for g and k.

(c) Using the formulas you chose for g and h, explain why all the values of g are positive, whereas all the values of h are negative.

t	$f(t)$	t	$g(t)$	t	$h(t)$	t	$k(t)$
6.0	−0.78	3.0	1.64	5.0	−2.46	3.0	0.64
6.5	−0.28	3.5	1.15	5.1	−2.43	3.5	0.15
7.0	0.16	4.0	0.74	5.2	−2.38	4.0	−0.26
7.5	0.44	4.5	0.52	5.3	−2.33	4.5	−0.48
8.0	0.49	5.0	0.54	5.4	−2.27	5.0	−0.46

38. The depth of water in a tank oscillates sinusoidally once every 6 hours. If the smallest depth is 5.5 feet and the largest depth is 8.5 feet, find a possible formula for the depth in terms of time in hours.

39. The voltage, V, of an electrical outlet in a home as a function of time, t (in seconds), is $V = V_0 \cos(120\pi t)$.

(a) What is the period of the oscillation?
(b) What does V_0 represent?
(c) Sketch the graph of V against t. Label the axes.

40. The power output, P, of a solar panel varies with the position of the sun. Let $P = 10 \sin\theta$ watts, where θ is the angle between the sun's rays and the panel, $0 \le \theta \le \pi$. On a typical summer day in Ann Arbor, Michigan, the sun rises at 6 am and sets at 8 pm and the angle is $\theta = \pi t/14$, where t is time in hours since 6 am and $0 \le t \le 14$.

(a) Write a formula for a function, $f(t)$, giving the power output of the solar panel (in watts) t hours after 6 am on a typical summer day in Ann Arbor.
(b) Graph the function $f(t)$ in part (a) for $0 \le t \le 14$.
(c) At what time is the power output greatest? What is the power output at this time?
(d) On a typical winter day in Ann Arbor, the sun rises at 8 am and sets at 5 pm. Write a formula for a function, $g(t)$, giving the power output of the solar panel (in watts) t hours after 8 am on a typical winter day.

41. A baseball hit at an angle of θ to the horizontal with initial velocity v_0 has horizontal range, R, given by

$$R = \frac{v_0^2}{g}\sin(2\theta).$$

Here g is the acceleration due to gravity. Sketch R as a function of θ for $0 \le \theta \le \pi/2$. What angle gives the maximum range? What is the maximum range?

42. The visitors' guide to St. Petersburg, Florida, contains the chart shown in Figure 1.62 to advertise their good weather. Fit a trigonometric function approximately to the data, where H is temperature in degrees Fahrenheit, and the independent variable is time in months. In order to do this, you will need to estimate the amplitude and period of the data, and when the maximum occurs. (There are many possible answers to this problem, depending on how you read the graph.)

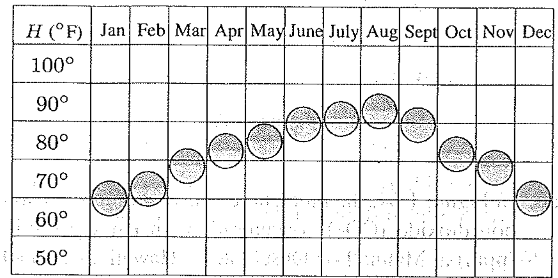

Figure 1.62: "St. Petersburg...where we're famous for our wonderful weather and year-round sunshine." (Reprinted with permission)

43. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, y meters, is given as a function of time, t, in hours since midnight by

$$y = D + A\cos\left(B(t - C)\right).$$

(a) What is the physical meaning of D?
(b) What is the value of A?
(c) What is the value of B? Assume the time between successive high tides is 12.4 hours.
(d) What is the physical meaning of C?

44. A compact disc spins at a rate of 200 to 500 revolutions per minute. What are the equivalent rates measured in radians per second?

45. When a car's engine makes less than about 200 revolutions per minute, it stalls. What is the period of the rotation of the engine when it is about to stall?

46. What is the period of the earth's revolution around the sun?

47. What is the approximate period of the moon's revolution around the earth?

48. For a boat to float in a tidal bay, the water must be at least 2.5 meters deep. The depth of water around the boat, $d(t)$, in meters, where t is measured in hours since midnight, is

$$d(t) = 5 + 4.6\sin(0.5t).$$

(a) What is the period of the tides in hours?
(b) If the boat leaves the bay at midday, what is the latest time it can return before the water becomes too shallow?

49. Match graphs A-D in Figure 1.63 with the functions below. Assume a, b, c and d are positive constants.

$$f(t) = \sin t + b \qquad h(t) = \sin t + e^{ct} + d$$
$$g(t) = \sin t + at + b \qquad r(t) = \sin t - e^{ct} + b$$

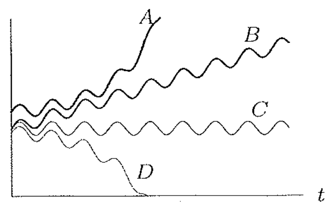

Figure 1.63

50. In Figure 1.64, the blue curve shows monthly mean carbon dioxide (CO_2) concentration, in parts per million (ppm) at Mauna Loa Observatory, Hawaii, as a function of t, in months, since December 2005. The black curve shows the monthly mean concentration adjusted for seasonal CO_2 variation.[35]

(a) Approximately how much did the monthly mean CO_2 increase between December 2005 and December 2010?

(b) Find the average monthly rate of increase of the monthly mean CO_2 between December 2005 and December 2010. Use this information to find a linear function that approximates the black curve.

(c) The seasonal CO_2 variation between December 2005 and December 2010 can be approximated by a sinusoidal function. What is the approximate period of the function? What is its amplitude? Give a formula for the function.

(d) The blue curve may be approximated by a function of the form $h(t) = f(t) + g(t)$, where $f(t)$ is sinusoidal and $g(t)$ is linear. Using your work in parts (b) and (c), find a possible formula for $h(t)$. Graph $h(t)$ using the scale in Figure 1.64.

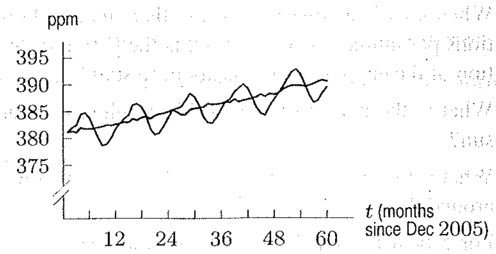

Figure 1.64

51. **(a)** Use a graphing calculator or computer to estimate the period of $2\sin\theta + 3\cos(2\theta)$.

(b) Explain your answer, given that the period of $\sin\theta$ is 2π and the period of $\cos(2\theta)$ is π.

52. Find the area of the trapezoidal cross-section of the irrigation canal shown in Figure 1.65.

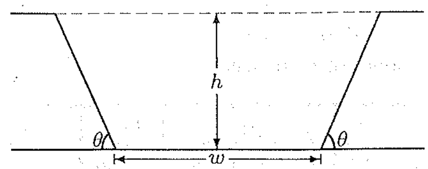

Figure 1.65

53. Graph $y = \sin x$, $y = 0.4$, and $y = -0.4$.

(a) From the graph, estimate to one decimal place all the solutions of $\sin x = 0.4$ with $-\pi \le x \le \pi$.

(b) Use a calculator to find $\arcsin(0.4)$. What is the relation between $\arcsin(0.4)$ and each of the solutions you found in part (a)?

(c) Estimate all the solutions to $\sin x = -0.4$ with $-\pi \le x \le \pi$ (again, to one decimal place).

(d) What is the relation between $\arcsin(0.4)$ and each of the solutions you found in part (c)?

54. Find the angle, in degrees, that a wheelchair ramp makes with the ground if the ramp rises 1 foot over a horizontal distance of

(a) 12 ft, the normal requirement[36]

(b) 8 ft, the steepest ramp legally permitted

(c) 20 ft, the recommendation if snow can be expected on the ramp

55. This problem introduces the arccosine function, or inverse cosine, denoted by $\boxed{\cos^{-1}}$ on most calculators.

(a) Using a calculator set in radians, make a table of values, to two decimal places, of $g(x) = \arccos x$, for $x = -1, -0.8, -0.6, \ldots, 0, \ldots, 0.6, 0.8, 1$.

(b) Sketch the graph of $g(x) = \arccos x$.

(c) Why is the domain of the arccosine the same as the domain of the arcsine?

(d) What is the range of the arccosine?

(e) Why is the range of the arccosine *not* the same as the range of the arcsine?

[35] http://www.esrl.noaa.gov/gmd/ccgg/trends/. Accessed March 2011. Monthly means joined by segments to highlight trends.

[36] http://www.access-board.gov/adaag/html/adaag.htm#4.1.6(3)a, accessed June 6, 2011.

Strengthen Your Understanding

In Problems 56–57, explain what is wrong with the statement.

56. For the function $f(x) = \sin(Bx)$ with $B > 0$, increasing the value of B increases the period.

57. For positive A, B, C, the maximum value of the function $y = A\sin(Bx) + C$ is $y = A$.

In Problems 58–59, give an example of:

58. A sine function with period 23.

59. A cosine function which oscillates between values of 1200 and 2000.

Are the statements in Problems 60–72 true or false? Give an explanation for your answer.

60. The function $f(\theta) = \cos\theta - \sin\theta$ is increasing on $0 \le \theta \le \pi/2$.

61. The function $f(t) = \sin(0.05\pi t)$ has period 0.05.

62. If t is in seconds, $g(t) = \cos(200\pi t)$ executes 100 cycles in one second.

63. The function $f(\theta) = \tan(\theta - \pi/2)$ is not defined at $\theta = \pi/2, 3\pi/2, 5\pi/2 \ldots$.

64. $\sin|x| = \sin x$ for $-2\pi < x < 2\pi$

65. $\sin|x| = |\sin x|$ for $-2\pi < x < 2\pi$

66. $\cos|x| = |\cos x|$ for $-2\pi < x < 2\pi$

67. $\cos|x| = \cos x$ for $-2\pi < x < 2\pi$

68. The function $f(x) = \sin(x^2)$ is periodic, with period 2π.

69. The function $g(\theta) = e^{\sin\theta}$ is periodic.

70. If $f(x)$ is a periodic function with period k, then $f(g(x))$ is periodic with period k for every function $g(x)$.

71. If $g(x)$ is a periodic function, then $f(g(x))$ is periodic for every function $f(x)$.

72. The function $f(x) = |\sin x|$ is even.

1.6 POWERS, POLYNOMIALS, AND RATIONAL FUNCTIONS

Power Functions

A *power function* is a function in which the dependent variable is proportional to a power of the independent variable:

> A **power function** has the form
>
> $f(x) = kx^p$, where k and p are constant.

For example, the volume, V, of a sphere of radius r is given by

$$V = g(r) = \frac{4}{3}\pi r^3.$$

As another example, the gravitational force, F, on a unit mass at a distance r from the center of the earth is given by Newton's Law of Gravitation, which says that, for some positive constant k,

$$F = \frac{k}{r^2} \quad \text{or} \quad F = kr^{-2}.$$

We consider the graphs of the power functions x^n, with n a positive integer. Figures 1.66 and 1.67 show that the graphs fall into two groups: odd and even powers. For n greater than 1, the odd powers have a "seat" at the origin and are increasing everywhere else. The even powers are first decreasing and then increasing. For large x, the higher the power of x, the faster the function climbs.