Math 107H

Topics for second exam

(Technically, everything covered on the first exam plus...)

Chapter 6: Exponentials, logarithms, and transcendental functions

Natural logarithms

Define $\ln x = \int_1^x dt/t$ = area under $1/t$ from 1 to x

$\ln x$ is a log; it turns products into sums: ln(ab) = ln(a) + ln(b)

$\ln(a^b) = b\ln(a)$; $\ln(a/b) = \ln(a) - \ln(b)$

$\frac{\mathrm{d}}{\mathrm{dx}}(\ln x) = 1/\mathrm{x}$; $\frac{\mathrm{d}}{\mathrm{dx}}(\ln(f(x))) = \frac{f'(x)}{f(x)}$ $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)|+\mathrm{c}$

Logarithmic differentiation: $f'(x) = f(x)\frac{\mathrm{d}}{\mathrm{dx}}(\ln(f(x)))$

useful for taking the derivative of products, powers, and quotients

Inverse functions and their derivatives

Basic idea: run a function backwards

y=f(x) ; 'assign' the value x to the input y ; x=g(y)

need g a function; so need f is one-to-one

f is one-to-one: if f(x)=f(y) then x=y ; if $x \neq y$ then $f(x) \neq f(y)$

$g = f^{-1}$, then $g(f(x)) = x$ and $f(g(x)) = x$ (i.e., $g \circ f$=Id and $f \circ g$=Id)

finding inverses: rewrite y=f(x) as x=some expression in y

graphs: if (a,b) on graph of f, then (b,a) on graph of f^{-1}

graph of f^{-1} is graph of f, reflected across line y=x

horizontal lines go to vertical lines; horizontal line test for inverse

derivative of the inverse: $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$

if $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$

The exponential function

e^x = inverse of $\ln x$; $e^{\ln x} = x$ (x >0), $\ln(e^x)$ = x (all x)

$e^{a+b} = e^a e^b$, $e^{ab} = (e^a)^b$; $e^1 = e = 2.718281828459045\ldots$

$\frac{\mathrm{d}}{\mathrm{dx}}(e^x) = e^x$; $\int e^x \,\mathrm{d}x = e^x$+c

a^x and $\log_a x$

$\ln(a^b)$ **should be** $b\ln a$, so $a^b = e^{b\ln a}$

$a^{b+c} = a^b a^c$; $a^{bc} = (a^b)^c$

$a^x = e^{x\ln a}$; $\frac{\mathrm{d}}{\mathrm{dx}}(a^x) = a^x \ln a$; $\int a^x \,\mathrm{d}x = \frac{a^x}{\ln a}$+c

$x^r = e^{r\ln x}$ (makes sense for *any real number* r) ; $\frac{\mathrm{d}}{\mathrm{dx}}(x^r) = rx^{r-1}$

$f(x)$=a^x is **either** always increasing ($a > 1$) **or** always decreasing ($a < 1$)

inverse is $g(x) = \log_a x = \frac{\ln x}{\ln a}$

Inverse trigonometric functions

Trig functions ($\sin x$, $\cos x$, $\tan x$, etc.) aren't one-to-one; make them!

$\sin x$, $-\pi/2 \leq x \leq \pi/2$ is one-to-one; inverse is Arcsin x

sin(Arcsin x)=x, all x; Arcsin($\sin x$)=x IF x in range above

$\tan x$, $-\pi/2 < x < \pi/2$ is one-to-one; inverse is Arctan x

tan(Arctan x)=x, all x; Arctan($\tan x$)=x IF x in range above

$\sec x$, $0 \leq x < \pi/2$ and $\pi/2 < x \leq \pi$, is one-to-one; inverse is Arcsec x

sec(Arcsec$x$)=x, all x; Arcsec($\sec x$)=x IF x in range above

Computing cos(Arcsin x), tan(Arcsec x), etc.; use right triangles

The other inverse trig functions aren't very useful,
they are essentially the negatives of the functions above.

Derivatives and integrals of inverse trig functions

Derivatives of inverse functions! Use right triangles to simplify.

$$\frac{d}{dx}(\text{ Arcsin } x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\text{ Arctan } x) = \frac{1}{x^2+1}$$

$$\frac{d}{dx}(\text{Arcsec } x) = \frac{1}{|x|\sqrt{x^2-1}}$$

Integrals: reverse the formulas.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \text{Arcsin}(\frac{x}{a}) + c$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\text{ Arctan}(\frac{x}{a}) + c$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{Arcsec}(|\frac{x}{a}|) + c$$

Chapter 7: Techniques of integration

Basic integration formulas (AKA dirty tricks)

u-substitution

$$\int f(g(x))g'(x)\ \mathrm{d}x = \int f(u)\ \mathrm{d}u\Big|_{u=g(x)}$$

complete the square

$$ax^2+bx+c = a(x^2+rx)+c = a(x+r/2)^2+(c-(r/2)^2)$$

Ex: $$\int \frac{1}{x^2+2x+2}\ \mathrm{d}x = \int \frac{1}{(x+1)^2+1}\ \mathrm{d}x$$

use trig identities

$\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sin(2x) = 2\sin x\cos x$, $\frac{\tan x}{\sec x}$=$\sin x$, etc.

Ex: $$\int \frac{\sin^2 x}{\cos x}\ \mathrm{d}x = \int \frac{1-\cos^2 x}{\cos x}\ \mathrm{d}x = \ldots$$

pull fractions apart; put fractions together!

Ex: $\displaystyle\int \frac{x+1}{x^3}\ \mathrm{d}x = \int x^{-2} + x^{-3}\ \mathrm{d}x = \ldots$

do poylnomial long division

Ex: $\displaystyle\int \frac{x^3}{x^2-1}\ \mathrm{d}x = \int x + \frac{x}{x^2-1}\ \mathrm{d}x = \ldots$

add zero, multiply by one

Ex: $\displaystyle\int \sec x\ \mathrm{d}x = \int \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}\ \mathrm{d}x = \ldots$

Integration by parts

Product rule: $\mathrm{d}(uv) = (\mathrm{d}u)v + u(\mathrm{d}v)$

reverse: $\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$

Ex: $\int x\cos x\ \mathrm{d}x$: set $u{=}x$, $\mathrm{d}v{=}\cos x\ \mathrm{d}x$

$\mathrm{d}u{=}\mathrm{d}x$, $v = \sin x$ (or any <u>other</u> antiderivative)

So: $\int x\cos x = x\sin x - \int \sin x\ \mathrm{d}x = \ldots$

special case: $\displaystyle\int f(x)\ \mathrm{d}x$; $u = f(x)$, $\mathrm{d}v{=}\mathrm{d}x$

$$\int f(x)\ \mathrm{d}x = xf(x) - \int xf'(x)\ \mathrm{d}x$$

Ex: $\displaystyle\int \text{Arcsin } x\ \mathrm{d}x = x \text{ Arcsin } x - \int \frac{x}{\sqrt{1-x^2}} = \ldots$

Trig substitution

Idea: get rid of square roots, by turning the stuff inside into a perfect square!

$\sqrt{a^2-x^2}$: set $x = a\sin u$. $\mathrm{d}x = a\cos u\ \mathrm{d}u$, $\sqrt{a^2-x^2} = a\cos u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\ \mathrm{d}x = \int \frac{\cos u}{\sin^2 u\cos u}\ \mathrm{d}u\Big|_{x=\sin u} = \ldots$

$\sqrt{a^2+x^2}$: set $x = a\tan u$. $\mathrm{d}x = a\sec^2 u\ \mathrm{d}u$, $\sqrt{a^2+x^2} = a\sec u$

Ex: $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\ \mathrm{d}x = \int \frac{2\sec^2 u}{(2\sec u)^3}\ \mathrm{d}u\Big|_{x=2\tan u} = \ldots$

$\sqrt{x^2-a^2}$: set $x = a\sec u$. $\mathrm{d}x = a\sec u\tan u\ \mathrm{d}u$, $\sqrt{x^2-a^2} = a\tan u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\ \mathrm{d}x = \int \frac{\sec u\tan u}{\sec^2 u\tan u}\ \mathrm{d}u\Big|_{x=\sec u} = \ldots$

Undoing the "u-substitution": use right triangles!

Trig integrals

What trig substitution leads to!

$$\int \sin^n x\ \cos^m x\ dx$$

If n is odd, keep one $\sin x$ and turn the others, in pairs, into $\cos x$ (using $\sin^2 x = 1 - \cos^2 x$), then do a u-substitution $u = \cos x$.

If m is odd, reverse the roles of $\sin x$ and $\cos x$.

If both are even, turn all of the $\sin x$ into $\cos x$ and use the double angle formula

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

This will convert $\cos^m x$ into a bunch of *lower powers* of $\cos(2x)$;
odd powers can be dealt with by substitution,
even powers by another application of the angle doubling formula!

$$\int \sec^n x \ \tan^m x \ dx = \int \frac{\sin^m x}{\cos^{n+m} x} \ dx$$

If n is *even*, set two of them aside and convert the rest to $\tan x$
using $\sec^2 x = \tan^2 x + 1$,
and use $u = \tan x$.
If m is *odd*, set one each of $\sec x$, $\tan x$ aside, convert the rest of the $\tan x$ to $\sec x$
using $\tan^2 x = \sec^2 x - 1$,
and use $u = \sec x$.

If n is odd and m is even, convert all of the $\tan x$ to $\sec x$,
leaving a bunch of powers of $\sec x$. Then use the *reduction formula*:

$$\int \sec^n x \ dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \ dx$$

At the end, reach $\int sec^2 x \ dx = \tan x + C$ or $\int \sec x \ dx = \ln|\sec x + \tan x| + C$

A little "trick" worth knowing:
the substitution $u = \frac{\pi}{2} - x$, since $\sin(\frac{\pi}{2} - x) = \cos x$ and $\cos(\frac{\pi}{2} - x) = \sin x$,
will *reverse* the roles of $\sin x$ and $\cos x$,
so, for example, will turn $\cot x$ into $\tan u$ and $\csc x$ into $\sec u$.
So, for example, the integral
$\int \frac{\cos^4 x}{\sin^7 x} \ dx = \int \csc^3 x \cot^4 x \ dx$, which our techniques don't cover,
becomes $\int \sec^3 u \tan^4 u \ du$, which our techniques <u>do</u> cover.