Math 107H
Topics for the third exam

(Technically, everything covered on the first two exams plus...)

Chapter 7: Techniques of integration

Partial fractions

rational function = quotient of polynomials
Idea: integrate by writing function as sum of simpler functions
Procedure: $f(x) = \frac{p(x)}{q(x)}$
(0): make sure degree(p)<degree(q); do long division if it isn't
(1): factor $q(x)$ into linear and irreducible quadratic factors
(2): group common factors together as powers
(3a): for each group $(x-a)^n$ add together: $\frac{a_1}{x-a} + \cdots + \frac{a_n}{(x-a)^n}$
(3b): for each group $(ax^2+bx+c)^n$ add together:
$\frac{a_1x+b_1}{ax^2+bx+c} + \cdots + \frac{a_nx+b_n}{(ax^2+bx+c)^n}$
(4) set $f(x)$ = sum; solve for the 'undetermined' coefficients
put sum over a common denomenator (=$q(x)$); set numerators equal.
always works: multiply out, group common powers, set coeffs of the two polynomials equal
Ex: $x+3 = a(x-1)+b(x-2) = (a+b)x+(-a-2b)$; $1=a+b$, $3=-a-2b$
linear term $(x-a)^n$: set $x=a$, will allow you to solve for a coefficient
if $n \geq 2$, take derivatives of both sides! set x=a, gives another coeff.
Ex: $\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} =$
$\frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)} = \ldots$

L'Hôpital's Rule

indeterminate forms: limits which 'evaluate' to 0/0 ; e.g. $\lim_{x\to 0} \frac{\sin x}{x}$
LR# 1: If $f(a)=g(a)=0$, f and g both differentiable near a, then
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
Other indeterminate forms: $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0
LR#2: if $f,g\to\infty$ as $x\to a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
Other cases: try to turn them into 0/0 or ∞/∞;
in the last three cases, do this by taking logs, first

Improper integrals

usual idea: $\int_a^b f(x)\,\mathrm{d}x = F(b)-F(a)$, where $F'(x)=f(x)$
Problems: $a=-\infty$, $b=\infty$; f blows up at a or b or somewhere in between
integral is "improper"; usual technique doesn't work. Solution to this:

$$\int_a^\infty f(x)\ \mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\ \mathrm{d}x \qquad \text{(similarly for } a=-\infty)$$

(blow up at a) $\displaystyle\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to a^-}\int_r^b f(x)\ \mathrm{d}x$ (similarly for blowup at b (or both!))

(blows up at c (b/w a and b)) $\displaystyle\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to c^-}\int_a^r f(x)\ \mathrm{d}x + \lim_{s\to c^+}\int_s^b f(x)\ \mathrm{d}x$

The integral <u>converges</u> if (all of the) limit(s) are finite

Comparison: $0 \le f(x) \le g(x)$ for all x;

if $\displaystyle\int_a^\infty g(x)\ \mathrm{d}x$ converges, so does $\displaystyle\int_a^\infty f(x)\ \mathrm{d}x$

Limit comparison: $f, g \ge 0$, $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$, $L \ne 0, \infty$, then

$\displaystyle\int_a^\infty f(x)\ \mathrm{d}x$ and $\displaystyle\int_a^\infty g(x)\ \mathrm{d}x$ either both converge or both diverge

Chapter 8: Infinite sequences and series

§1: Limits of sequences of numbers

A sequence is: a string of numbers; a function $f{:}\mathbf{N}\to\mathbf{R}$; write $f(n) = a_n$

$a_n = n$-th term of the sequence

Basic question: convergence/divergence

$\displaystyle\lim_{n\to\infty} a_n = L$ (or $a_n \to L$) if

eventually all of the a_n are always as close to L as we like, i.e.

for any $\epsilon > 0$, there is an N so that if $n \ge N$ then $|a_n - L| < \epsilon$

Ex.: $a_n = 1/n$ converges to 0 ; can always choose $N{=}1/\epsilon$

$a_n = (-1)^n$ diverges; terms of the sequence never settle down to a <u>single</u> number

If a_n is increasing ($a_{n+1} \ge a_n$ for every n) and bounded from above ($a_n \le M$ for every n, for some M) , then a_n converges (but not necessarily to M !)

limit is smallest number bigger than all of the terms of the sequence

Limit theorems for sequences

Idea: limits of sequences are a lot like limits of functions

If $a_n \to L$ and $b_n \to M$, then

$(a_n + b_n \to L + M \qquad (a_n - b_n) \to L - M \qquad (a_n b_n) \to LM$, and

$(a_n/b_n) \to L/M$ (provided M, all b_n are $\ne 0$)

Sqeeze play theorem: if $a_n \le b_n \le c_n$ (for all n large enough) and

$a_n \to L$ and $c_n \to L$, then $b_n \to L$

If $a_n \to L$ and $f{:}\mathbf{R}\to\mathbf{R}$ is continuous at L, then $f(a_n) \to f(L)$

if $a_n = f(n)$ for some function $f{:}\mathbf{R}\to\mathbf{R}$ and $\displaystyle\lim_{x\to\infty} f(x) = L$, then $a_n \to L$

(allows us to use L'Hopital's Rule!)

Another basic list: (x = fixed number, k = konstant)

$\dfrac{1}{n} \to 0 \qquad k \to k \qquad x^{\frac{1}{n}} \to 1$

$n^{\frac{1}{n}} \to 1 \qquad (1 + \dfrac{x}{n})^n \to e^x \qquad \dfrac{x^n}{n!} \to 0$

$x^n \to \{\ 0$, if $|x| < 1$; 1, if $x = 1$; diverges, otherwise $\}$

Infinite series

An infinite series is an infinite sum of numbers

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n \qquad \text{(summation notation)}$$

n-th term of series $= a_n$; N-th partial sum of series $= s_N = \sum_{n=1}^{N} a_n$

An infinite series **converges** is the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges

We may start the series anywhere: $\sum_{n=0}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=3437}^{\infty} a_n$, etc. ;

convergence is unaffected (but the number it adds up to is!)

Ex. geometric series: $a_n = ar^n$; $\sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$

if $|r| < 1$; otherwise, the series **diverges**.

Ex. Telescoping series: partial sums s_N 'collapse' to a simple expression

E.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})$; $s_N = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - (\frac{1}{N+1} + \frac{1}{N+2}))$

n-th term test: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

So if the n-th terms **don't** go to 0, then $\sum_{n=1}^{\infty} a_n$ diverges

Basic limit theorems: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \qquad \sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty}(ka_n) = k\sum_{n=1}^{\infty} a_n$$

Truncating a series: $\sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \sum_{n=1}^{N-1} a_n$

Comparison tests

Again, think $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ all n

Convergence depends only on partial sums s_N being **bounded**

One easy way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all n (past a certain point), then

if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$; if $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: a_n and $b_n \geq 0$ for all n, $\dfrac{a_n}{b_n} \to L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ anf $\sum b_n$ either **both** converge or **both** diverge

If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$

If $L = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$

(Why? eventually $(L/2)b_n \leq a_n \leq (3L/2)b_n$; so can use comparison test.)

Ex: $\sum 1/(n^3 - 1)$ converges; L-comp with $\sum 1/n^3$

$\sum n/3^n$ converges; L-comp with $\sum 1/2^n$

$\sum 1/(n \ln(n^2+1)$ diverges; L-comp with $\sum 1/(n \ln n)$

The integral test

Idea: $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ all n, then the partial sums

$\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence;

so converges exactly when bounded from above

If (eventually) $a_n = f(n)$ for a **decreasing** function $f : [a, \infty) \to \mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x)\ \mathrm{d}x \leq s_N = \sum_{n=a}^{N} a_n \leq \int_a^N f(x)\ \mathrm{d}x$$

so $\displaystyle\sum_{n=a}^{\infty} a_n$ converges exactly when $\displaystyle\int_a^{\infty} f(x)\ \mathrm{d}x$ converges

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$ (p-series)

The ratio and root tests

A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

If $\sum |a_n|$ converges then $\sum a_n$ converges

Previous tests have you compare your series with **something else** (another series, an improper integral); these tests compare a series with itself (sort of)

Ratio Test: $\sum a_n$, $a_n \neq 0$ all n; $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Root Test: $\sum a_n$, $\displaystyle\lim_{n\to\infty} |a_n|^{1/n} = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Ex: $\sum \frac{4^n}{n!}$ converges by the ratio test $\quad \sum \frac{n^5}{n^n}$ converges by the root test

Power series

Idea: turn a series into a function, by making the terms a_n ˘depend on x
replace a_n with $a_n x^n$; series of powers

$\sum_{n=0}^{\infty} a_n x^n$ = power series centered at 0

$$\sum_{n=0}^{\infty} a_n (x-a)^n = \text{power series centered at } a$$

Big question: for what x does it converge? Solution from ratio test

$\lim \left|\frac{a_{n+1}}{a_n}\right| = L$, set $R = \frac{1}{L}$

then $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges absolutely for $|x-a| < R$

diverges for $|x-a| > R$; $\quad R$ = radius of convergence

Ex.: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; conv. for $|x| < 1$

Why care about power series?

Idea: partial sums $\sum_{k=0}^{n} a_k x^k$ are ˇpolynomials;

if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then the poly's make good approximations for f

Differentiation and integration of power series

Idea: if you diff. or int. each term of a power series, you get a power series which is the deriv. or integral of the original one.

If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of conv R,

then so does $g(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$, and $g(x) = f'(x)$

and so does $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$, and $g'(x) = f(x)$

Ex: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $\quad f'(x) = f(x)$, so (since $f(0) = 1$) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ex.: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $\quad -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad$ (for $|x| < 1$)

Ex:. $\arctan x = \int \frac{1}{1-(-x^2)}\, dx = \int \sum_{n=0}^{\infty} (-x^2)^n\, dx =$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \text{(for } |x| < 1\text{)}$$

Taylor series

Idea: start with function $f(x)$, find power series <u>for</u> it.

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, then (term by term diff.)

$$f^{(n)}(a) = n!a_n \text{ ; SO } a_n = \frac{f^{(n)}(a)}{n!}$$

Starting with f, define $P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$,

the Taylor series for f, centered at a.

$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$, the n-th Taylor polynomial for f.

Ex.: $f(x) = \sin x$, then $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

Big questions: Is $f(x) = P(x)$? (I.e., does $f(x) - P_n(x)$ tend to 0 ?)
If so, how well do the P_n's approximate f ? (I.e., how small is $f(x) - P_n(x)$?)

Error estimates

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

means that the value of f at a point x (far from a) can be determined just from the behavior of f near a (i.e., from the derivs. of f at a). This is a very powerful property, one that we wouldn't ordinarily expect to be true. The amazing thing is that it often is:

$$P(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ ; } P_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(k-a)^n \text{ ;}$$

$R_n(x,a)$= $f(x) - P_n(x,a)$ = n-th remainder term = error in using P_n to approximate f

Taylor's remainder theorem : estimates the size of $R_n(x,a)$

If $f(x)$ and all of its derivatives (up to $n+1$) are continuous on $[a,b]$, then

$f(b) = P_n(b,a) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$, for some c in $[a,b]$

i.e., for each x, $R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some c between a and x

so if $|F^{(n+1)}(x)||leqM$ for every x in $[a,b]$, then $|R_n(x,a)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$ for every x in $[a,b]$

Ex.: $f(x)$=$\sin x$, then $|f^{(n+1)}(x)| \leq 1$ for all x, so $|R_n(x,0)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

so $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

Similarly, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

Use Taylor's remainder to estimate values of functions:

$$e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}, \text{ so } e{=}e^1{=}\sum_{n=0}^{\infty} \frac{1}{(n)!}$$

$$|R_n(1,0)| = \frac{f^{(n+1)}(c)}{(n+1)!} = \frac{e^c}{(n+1)!} \leq \frac{e^1}{(n+1)!} \leq \frac{4}{(n+1)!}$$

since $e < 4$ (since $\ln(4) > (1/2)(1) + (1/4)(2) = 1$)

(Riemann sum for integral of $1/x$)

so since $\dfrac{4}{(13+1)!} = 4.58{\times}10^{-11}$,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{13!}$$, to 10 decimal places.

Other uses: if you know the Taylor series, it tells you the values of the derivatives at the center.

Ex.: $e^x{=}\sum_{n=0}^{\infty} \dfrac{(x)^n}{(n)!}$, so

$xe^x = \sum_{n=0}^{\infty} \dfrac{(x)^{n+1}}{(n)!}$, so

15th deriv of xe^x , at 0, is 15!(coeff of x^{15}) $= \dfrac{15!}{14!} = 15$

Substitutions: new Taylor series out of old ones

$$\text{Ex. } \sin^2 x = \frac{1-\cos(2x)}{2} = \frac{1}{2}(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= \frac{1}{2}(1 - (1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= \tfrac{2x^2}{2!} - \tfrac{2^3x^4}{4!} + \tfrac{2^5x^6}{6!} - \tfrac{2^7x^8}{8!} + \cdots$$

Integrate functions we can't handle any other way:

Ex.: $e^{x^2} = \sum_{n=0}^{\infty} \dfrac{(x)^2 n}{(n)!}$, so

$$\int e^{x^2} \, \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(x)^{2n+1}}{n!(2n+1)}$$