**Math 107H**

**Topics since the third exam**

**Chapter 9:** Parametric curves

The motivation: think of the graph of $y = f(x)$ as a path that we are walking along. The 'right' way to think of this is that we are visiting each point of the graph at various times $t$, e.g.,

$$x = t \ , \ y = f(x) = f(t)$$

But we need not be limited to having $x = t$ ; we can more generally describe our path as

$$x = x(t) \ , \ y = y(t)$$

This is a *parametric curve*; it describes a curve in the plane, and how we traverse it through time. The advantage is that the curve we describe need not be the graph of a function. $t$ = the parameter = the independent variable ; $x$ and $y$ = dependent variables

A circle of radius 1 centered at $(0,0)$ : $x^2 + y^2 = 1$

$$x(t) = \cos t \ , \ y(t) = \sin t \qquad 0 \leq t \leq 2\pi$$

Twice as fast around: $x(t) = \cos 2t \ , \ y(t) = \sin 2t \qquad 0 \leq t \leq \pi$

A circle of radius $r$ centered at $(a,b)$ : $(x-a)^2 + (y-b)^2 = r^2$

Think: $x - a = r\cos t \ , \ y - b = r \sin t$

$$x(t) = a + r\cos t \ , \ y(t) = b + r\sin t \qquad 0 \leq t \leq 2\pi$$

An ellipse: $(x/a)^2 + (y/b)^2 = 1$

$$x(t) = a\cos t \ , \ y(t) = b\sin t \qquad 0 \leq t \leq 2\pi$$

A line through $(a,b)$ and $(c,d)$

$$x(t) = a + t(c-a) \ , \ y(t) = b + t(d-b)$$

Finding an $(x,y)$ equation from a parametric equation: (if possible) solve for $x = x(t)$ or $y = y(t)$ as $t$ = expression in $x$ or $y$, then plug into the other equation.

Ex: $x = t^2 - 1$ , $y = t^3 + t - 1$ , then $x + 1 = t^2$ so $t = \pm\sqrt{x+1}$, so $y = (\pm\sqrt{x+1})^3 + (\pm\sqrt{x+1}) - 1$

**Calculus of curves**

Thinking of a parametric curve as a path that we are traversing, we are at each instant aware of (at least) two things: how fast we are going and what direction we are going. Each can be computed essentially as we would for a graph.

Speed = the limit of (distance)/(time interval) as the time interval shrinks to 0.

average speed = $\sqrt{(\Delta x)^2 + (\Delta y)^2}/\Delta t = \sqrt{(\Delta x/\Delta t)^2 + (\Delta y/\Delta t)^2}$

instantaneous speed = $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(x'(t))^2 + (y'(t))^2}$

direction = slope of tangent line - limit of slopes of secant lines

secant lines: slope = $\Delta y/\Delta x = (\Delta y/\Delta t)/(\Delta x/\Delta t)$

tangent lines: slope = $(dy/dt)/(dx/dt) = y'(t)/x'(t)$

We can encode both of these in the *velocity vector* $(x'(t), y'(t))$

A parametric curve $x = x(t)$ , $y = y(t)$ , $a \leq t \leq b$ with $x(a) = x(b)$ , $y(a) = y(b)$ ends where it begins; it is a *closed curve.* Such a curve surrounds and encloses a region $R$ in the plane.

If the curve goes around the region counterclockwise, then the area of the region can be computed as

$$\text{Area} = \int_a^b x(t)y'(t)\ dt = -\int_a^b y(t)x'(t)\ dt$$

We will see why this formula is true in Math 208....

**Arclength and surface area**

Just as with graphs of functions, we can compute the length of a paramentric curve and the surface area when a curve is rotated around an axis:

Length: we approximate it the same way, as a sum of lengths of line seqments that approximate the curve. Each segment has length

$$\sqrt{(\Delta x)^2 + (\Delta y/)^2} = \sqrt{(\Delta x/\Delta t)^2 + (\Delta y/\Delta t)^2}\Delta t \approx \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

so the length of the curve is $\displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$

Surface area: if we spin the curve $x = x(t)$ , $y = y(t)$ , $a \leq t \leq b$ around the line $y = c$ then, just like before, we can approximate the surface by frustra of cones, each having area approximately

$2\pi|y(t) - c|\sqrt{(x'(t))^2 + (y'(t))^2}\ dt = (2\pi)(\text{radius})(\text{length})$

and so the area of the surface of revolution is

$$2\pi\int_a^b |y(t) - c|\sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

Ex: for the ellipse $x = 3\cos t$ , $y = 5\sin t$ , $0 \leq t \leq 2\pi$, spun around $y = 7$, we have

$\text{Area} = 2\pi\displaystyle\int_0^{2\pi} (7 - 3\sin t)\sqrt{9\sin^2 t + 25\cos^2 t}\ dt = 2\pi\int_0^{2\pi} (7 - 3\sin t)\sqrt{9 + 16\cos^2 t}\ dt$