

Math 107H
Topics for the third exam

(Technically, everything covered on the first two exams plus...)

Chapter 7: Techniques of integration

Partial fractions

rational function = quotient of polynomials

Idea: integrate by writing function as sum of simpler functions

Procedure: $f(x) = \frac{p(x)}{q(x)}$

(0): make sure $\text{degree}(p) < \text{degree}(q)$; do long division if it isn't

(1): factor $q(x)$ into linear and irreducible quadratic factors

(2): group common factors together as powers

(3a): for each group $(x - a)^n$ add together: $\frac{a_1}{x - a} + \dots + \frac{a_n}{(x - a)^n}$

(3b): for each group $(ax^2 + bx + c)^n$ add together:

$$\frac{a_1x + b_1}{ax^2 + bx + c} + \dots + \frac{a_nx + b_n}{(ax^2 + bx + c)^n}$$

(4) set $f(x) = \text{sum}$; solve for the 'undetermined' coefficients

put sum over a common denominator ($=q(x)$); set numerators equal.

always works: multiply out, group common powers, set coeffs of the two polynomials equal

Ex: $x + 3 = a(x - 1) + b(x - 2) = (a + b)x + (-a - 2b)$; $1 = a + b$, $3 = -a - 2b$

linear term $(x - a)^n$: set $x = a$, will allow you to solve for a coefficient

if $n \geq 2$, take derivatives of both sides! set $x=a$, gives another coeff.

$$\text{Ex: } \frac{x^2}{(x - 1)^2(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 1} =$$

$$\frac{A(x - 1)(x^2 + 1) + B(x^2 + 1) + (Cx + D)(x - 1)^2}{(x - 1)^2(x^2 + 1)} = \dots$$

L'Hôpital's Rule

indeterminate forms: limits which 'evaluate' to $0/0$; e.g. $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

LR# 1: If $f(a) = g(a) = 0$, f and g both differentiable near a , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Other indeterminate forms: $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0 , 1^∞ , ∞^0

LR#2: if $f, g \rightarrow \infty$ as $x \rightarrow a$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Other cases: try to turn them into $0/0$ or ∞/∞ ;

in the last three cases, do this by taking logs, first

Improper integrals

usual idea: $\int_a^b f(x) dx = F(b) - F(a)$, where $F'(x) = f(x)$

Problems: $a = -\infty$, $b = \infty$; f blows up at a or b or somewhere in between
integral is "improper"; usual technique doesn't work. Solution to this:

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx \quad (\text{similarly for } a = -\infty)$$

(blow up at a) $\int_a^b f(x) dx = \lim_{r \rightarrow a^-} \int_r^b f(x) dx$ (similarly for blowup at b (or both!))

(blows up at c (b/w a and b)) $\int_a^b f(x) dx = \lim_{r \rightarrow c^-} \int_a^r f(x) dx + \lim_{s \rightarrow c^+} \int_s^b f(x) dx$

The integral converges if (all of the) limit(s) are finite

Comparison: $0 \leq f(x) \leq g(x)$ for all x ;

$$\text{if } \int_a^\infty g(x) dx \text{ converges, so does } \int_a^\infty f(x) dx$$

Limit comparison: $f, g \geq 0$, $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$, $L \neq 0, \infty$, then

$$\int_a^\infty f(x) dx \text{ and } \int_a^\infty g(x) dx \text{ either both converge or both diverge}$$

Chapter 8: Infinite sequences and series

§1: Limits of sequences of numbers

A sequence is: a string of numbers; a function $f: \mathbf{N} \rightarrow \mathbf{R}$; write $f(n) = a_n$
 $a_n = n$ -th term of the sequence

Basic question: convergence/divergence

$$\lim_{n \rightarrow \infty} a_n = L \text{ (or } a_n \rightarrow L) \text{ if}$$

eventually all of the a_n are always as close to L as we like, i.e.

for any $\epsilon > 0$, there is an N so that if $n \geq N$ then $|a_n - L| < \epsilon$

Ex.: $a_n = 1/n$ converges to 0 ; can always choose $N=1/\epsilon$

$a_n = (-1)^n$ diverges; terms of the sequence never settle down to a single number

If a_n is increasing ($a_{n+1} \geq a_n$ for every n) and bounded from above

($a_n \leq M$ for every n , for some M), then a_n converges (but not necessarily to M !)

limit is smallest number bigger than all of the terms of the sequence

Limit theorems for sequences

Idea: limits of sequences are a lot like limits of functions

If $a_n \rightarrow L$ and $b_n \rightarrow M$, then

$$(a_n + b_n \rightarrow L + M \quad (a_n - b_n) \rightarrow L - M \quad (a_n b_n) \rightarrow LM, \text{ and}$$

$$(a_n/b_n) \rightarrow L/M \text{ (provided } M, \text{ all } b_n \text{ are } \neq 0)$$

Squeeze play theorem: if $a_n \leq b_n \leq c_n$ (for all n large enough) and

$$a_n \rightarrow L \text{ and } c_n \rightarrow L, \text{ then } b_n \rightarrow L$$

If $a_n \rightarrow L$ and $f: \mathbf{R} \rightarrow \mathbf{R}$ is continuous at L , then $f(a_n) \rightarrow f(L)$

if $a_n = f(n)$ for some function $f: \mathbf{R} \rightarrow \mathbf{R}$ and $\lim_{x \rightarrow \infty} f(x) = L$, then $a_n \rightarrow L$

(allows us to use L'Hopital's Rule!)

Another basic list: ($x =$ fixed number, $k =$ konstant)

$$\frac{1}{n} \rightarrow 0 \quad k \rightarrow k \quad x^{\frac{1}{n}} \rightarrow 1$$

$$n^{\frac{1}{n}} \rightarrow 1 \quad \left(1 + \frac{x}{n}\right)^n \rightarrow e^x \quad \frac{x^n}{n!} \rightarrow 0$$

$$x^n \rightarrow \{ 0, \text{ if } |x| < 1 ; 1, \text{ if } x = 1 ; \text{ diverges, otherwise } \}$$

Infinite series

An infinite series is an infinite sum of numbers

$$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n \quad (\text{summation notation})$$

$$n\text{-th term of series} = a_n ; N\text{-th partial sum of series} = s_N = \sum_{n=1}^N a_n$$

An infinite series **converges** is the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges

We may start the series anywhere: $\sum_{n=0}^{\infty} a_n, \sum_{n=1}^{\infty} a_n, \sum_{n=3437}^{\infty} a_n, \text{ etc. ;}$

convergence is unaffected (but the number it adds up to is!)

Ex. geometric series: $a_n = ar^n ; \sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$

if $|r| < 1$; otherwise, the series **diverges**.

Ex. Telescoping series: partial sums s_N 'collapse' to a simple expression

E.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right); s_N = \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} - \left(\frac{1}{N+1} + \frac{1}{N+2} \right) \right)$

n -th term test: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$

So if the n -th terms **don't** go to 0, then $\sum_{n=1}^{\infty} a_n$ diverges

Basic limit theorems: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n + b_n) &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n & \sum_{n=1}^{\infty} (a_n - b_n) &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \\ \sum_{n=1}^{\infty} (ka_n) &= k \sum_{n=1}^{\infty} a_n \end{aligned}$$

Truncating a series: $\sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \sum_{n=1}^{N-1} a_n$

Comparison tests

Again, think $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ all n

Convergence depends only on partial sums s_N being **bounded**
One easy way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all n (past a certain point), then

$$\text{if } \sum_{n=1}^{\infty} b_n \text{ converges, so does } \sum_{n=1}^{\infty} a_n \quad ; \quad \text{if } \sum_{n=1}^{\infty} a_n \text{ diverges, so does } \sum_{n=1}^{\infty} b_n$$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: a_n and $b_n \geq 0$ for all n , $\frac{a_n}{b_n} \rightarrow L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ and $\sum b_n$ either **both** converge or **both** diverge

If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$

If $L = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$

(Why? eventually $(L/2)b_n \leq a_n \leq (3L/2)b_n$; so can use comparison test.)

Ex: $\sum 1/(n^3 - 1)$ converges; L-comp with $\sum 1/n^3$

$\sum n/3^n$ converges; L-comp with $\sum 1/2^n$

$\sum 1/(n \ln(n^2 + 1))$ diverges; L-comp with $\sum 1/(n \ln n)$

The integral test

Idea: $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ all n , then the partial sums

$\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence;

so converges exactly when bounded from above

If (eventually) $a_n = f(n)$ for a **decreasing** function $f : [a, \infty) \rightarrow \mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x) dx \leq s_N = \sum_{n=a}^N a_n \leq \int_a^N f(x) dx$$

so $\sum_{n=a}^{\infty} a_n$ converges exactly when $\int_a^{\infty} f(x) dx$ converges

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$ (p -series)

The ratio and root tests

A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

If $\sum |a_n|$ converges then $\sum a_n$ converges

Previous tests have you compare your series with **something else** (another series, an improper integral); these tests compare a series with itself (sort of)

Ratio Test: $\sum a_n$, $a_n \neq 0$ all n ; $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Root Test: $\sum a_n$, $\lim_{n \rightarrow \infty} |a_n|^{1/n} = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Ex: $\sum \frac{4^n}{n!}$ converges by the ratio test $\sum \frac{n^5}{n^n}$ converges by the root test

Power series

Idea: turn a series into a function, by making the terms a_n depend on x
 replace a_n with $a_n x^n$; series of powers

$\sum_{n=0}^{\infty} a_n x^n =$ power series centered at 0

$\sum_{n=0}^{\infty} a_n (x-a)^n =$ power series centered at a

Big question: for what x does it converge? Solution from ratio test

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = L, \text{ set } R = \frac{1}{L}$$

then $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges absolutely for $|x-a| < R$

diverges for $|x-a| > R$; $R =$ radius of convergence

Ex.: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; conv. for $|x| < 1$

Why care about power series?

Idea: partial sums $\sum_{k=0}^n a_k x^k$ are polynomials;

if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then the poly's make good approximations for f

Differentiation and integration of power series

Idea: if you diff. or int. each term of a power series, you get a power series which is the deriv. or integral of the original one.

If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of conv R ,

then so does $g(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$, and $g(x) = f'(x)$

and so does $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$, and $g'(x) = f(x)$

Ex: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $f'(x) = f(x)$, so (since $f(0) = 1$) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ex.: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ (for $|x| < 1$)

Ex.: $\arctan x = \int \frac{1}{1-(-x^2)} dx = \int \sum_{n=0}^{\infty} (-x^2)^n dx =$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad (\text{for } |x| < 1)$$

Taylor series

Idea: start with function $f(x)$, find power series for it.

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, then (term by term diff.)

$$f^{(n)}(a) = n!a_n ; \text{ SO } a_n = \frac{f^{(n)}(a)}{n!}$$

Starting with f , define $P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$,

the Taylor series for f , centered at a .

$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k$, the n -th Taylor polynomial for f .

Ex.: $f(x) = \sin x$, then $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

Big questions: Is $f(x) = P(x)$? (I.e., does $f(x) - P_n(x)$ tend to 0?)

If so, how well do the P_n 's approximate f ? (I.e., how small is $f(x) - P_n(x)$?)

Error estimates

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

means that the value of f at a point x (far from a) can be determined just from the behavior of f near a (i.e., from the derivs. of f at a). This is a very powerful property, one that we wouldn't ordinarily expect to be true. The amazing thing is that it often is:

$$P(x, a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n ; P_n(x, a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k ;$$

$R_n(x, a) = f(x) - P_n(x, a) = n$ -th remainder term = error in using P_n to approximate f

Taylor's remainder theorem : estimates the size of $R_n(x, a)$

If $f(x)$ and all of its derivatives (up to $n+1$) are continuous on $[a, b]$, then

$$f(b) = P_n(b, a) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}, \text{ for some } c \text{ in } [a, b]$$

i.e., for each x , $R_n(x, a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some c between a and x

so if $|f^{(n+1)}(x)| \leq M$ for every x in $[a, b]$, then $|R_n(x, a)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$
for every x in $[a, b]$

Ex.: $f(x) = \sin x$, then $|f^{(n+1)}(x)| \leq 1$ for all x , so $|R_n(x, 0)| \leq \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0$ as $n \rightarrow \infty$

$$\text{so } \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Similarly, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

Use Taylor's remainder to estimate values of functions:

$$e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}, \text{ so } e=e^1 = \sum_{n=0}^{\infty} \frac{1}{(n)!}$$

$$|R_n(1, 0)| = \frac{f^{(n+1)}(c)}{(n+1)!} = \frac{e^c}{(n+1)!} \leq \frac{e^1}{(n+1)!} \leq \frac{4}{(n+1)!}$$

since $e < 4$ (since $\ln(4) > (1/2)(1) + (1/4)(2) = 1$)
(Riemann sum for integral of $1/x$)

$$\text{so since } \frac{4}{(13+1)!} = 4.58 \times 10^{-11},$$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots + \frac{1}{13!}, \text{ to 10 decimal places.}$$

Other uses: if you know the Taylor series, it tells you the values of the derivatives at the center.

$$\text{Ex.: } e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}, \text{ so}$$

$$xe^x = \sum_{n=0}^{\infty} \frac{(x)^{n+1}}{(n)!}, \text{ so}$$

$$15\text{th deriv of } xe^x, \text{ at } 0, \text{ is } 15!(\text{coeff of } x^{15}) = \frac{15!}{14!} = 15$$

Substitutions: new Taylor series out of old ones

$$\text{Ex. } \sin^2 x = \frac{1 - \cos(2x)}{2} = \frac{1}{2} \left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right)$$

$$= \frac{1}{2} \left(1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \right) \right)$$

$$= \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \frac{2^7 x^8}{8!} + \dots$$

Integrate functions we can't handle any other way:

$$\text{Ex.: } e^{x^2} = \sum_{n=0}^{\infty} \frac{(x)^{2n}}{(n)!}, \text{ so}$$

$$\int e^{x^2} dx = \sum_{n=0}^{\infty} \frac{(x)^{2n+1}}{n!(2n+1)}$$