Now that we know how to find antiderivatives for a number of functions, we return to the problem of the plummeting space shuttle that opened the section.

Example 1.13 Finding the Position of a Falling Object Given Its Acceleration

If a space shuttle's downward acceleration is given by $y''(t) = -32 \text{ ft/s}^2$, find the position function $y(t)$. Assume that the shuttle's initial velocity is $y'(0) = -100$ ft/s, and that its initial position is $y(0) = 100{,}000$ feet.

Solution We have to undo two derivatives, so we compute two antiderivatives. First, we have

$$y'(t) = \int y''(t)\,dt = \int (-32)\,dt = -32t + c.$$

Recall that $y'(t)$ is the velocity of the shuttle, given in units of feet per second. We can evaluate the constant c using the given initial velocity. Since

$$v(t) = y'(t) = -32t + c$$

and $v(0) = -100$, we must have

$$-100 = v(0) = -32(0) + c = c,$$

so that $c = -100$. Thus, the velocity is $y'(t) = -32t - 100$. Next, we have

$$\begin{aligned} y(t) &= \int y'(t)\,dt = \int (-32t - 100)\,dt = -32\left(\tfrac{1}{2}t^2\right) - 100t + c \\ &= -16t^2 - 100t + c. \end{aligned}$$

Recall that $y(t)$ gives the height of the shuttle, measured in feet. Using the initial position, we have

$$100{,}000 = y(0) = -16(0) - 100(0) + c = c.$$

Thus, $c = 100{,}000$ and

$$y(t) = -16t^2 - 100t + 100{,}000.$$

Keep in mind that this models the space shuttle's altitude assuming that the only force acting on the shuttle is gravity (i.e., there is no air drag or lift). In this case, we have a disastrous (and fortunately, unrealistic) outcome. (Compute the velocity of the shuttle on landing, to see why.)

■

EXERCISES

1. In the text, we emphasized that the indefinite integral represents *all* antiderivatives of a given function. To understand why this is important, consider a situation where you know the net force, $F(t)$, acting on an object. By Newton's second law, $F = ma$. For the position function $s(t)$, this translates to $a(t) = s''(t) = F(t)/m$. To compute $s(t)$, you need to compute an antiderivative of the force function $F(t)/m$ followed by an antiderivative of the first antiderivative. But, suppose you were unable to find *all* antiderivatives. How would you know whether you had computed the antiderivative that corresponds to the

position function? In physical terms, explain why it is reasonable to expect that there is only one antiderivative corresponding to a given set of initial conditions.

2. In the text, we presented a one-dimensional model of a space shuttle flight. We ignored some of the forces on the shuttle so that the resulting mathematical equation would be one that we could solve. You may wonder what the benefit of doing this is. Weigh the relative worth of having an unsolvable but realistic model versus having a solution of a model that is only partially accurate. Keep in mind that when you toss trash into a wastebasket you do not take the curvature of the earth into account.

3. Verify that $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$ and $\int xe^x\,dx = xe^x - e^x + c$ by computing derivatives of the proposed antiderivatives. Which derivative rules did you use? Why does this make it unlikely that we will find a general product (antiderivative) rule for $\int f(x)g(x)\,dx$?

4. We stated in the text that we do not yet have a formula for the antiderivative of several elementary functions, including $\ln x$, $\sec x$ and $\csc x$. Given a function $f(x)$, explain what determines whether or not we have a simple formula for $\int f(x)\,dx$. For example, why is there a simple formula for $\int \sec x \tan x\,dx$ but not $\int \sec x\,dx$?

In exercises 5–40, find the general antiderivative.

5. $\int 3x^4\,dx$

6. $\int 5x^2\,dx$

7. $\int (3x^4 - 3x)\,dx$

8. $\int (x^3 - 2)\,dx$

9. $\int 3\sqrt{x}\,dx$

10. $\int (4x - 2\sqrt{x})\,dx$

11. $\int \left(3 - \frac{1}{x^4}\right)dx$

12. $\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right)dx$

13. $\int \frac{x^{1/3} - 3}{x^{2/3}}\,dx$

14. $\int \frac{x + 2x^{3/4}}{x^{5/4}}\,dx$

15. $\int (2\sin x + \cos x)\,dx$

16. $\int (3\cos x - \sin x)\,dx$

17. $\int 2\sec x \tan x\,dx$

18. $\int 4\csc^2 x\,dx$

19. $\int 5\sec^2 x\,dx$

20. $\int 4\csc x \cot x\,dx$

21. $\int (3e^x - 2)\,dx$

22. $\int (4x - 2e^x)\,dx$

23. $\int (3\cos x - 1/x)\,dx$

24. $\int (2x^{-1} + \sin x)\,dx$

25. $\int \frac{4x}{x^2 + 4}\,dx$

26. $\int \frac{2x^3}{x^4 + 1}\,dx$

27. $\int \left(5x - \frac{3}{e^x}\right)dx$

28. $\int (2\cos x - \sqrt{e^x})\,dx$

29. $\int 5\sin 2x\,dx$

30. $\int 4\cos 5x\,dx$

31. $\int (e^{3x} - x)\,dx$

32. $\int (4\sin 3x - 1)\,dx$

33. $\int 3\sec 2x \tan 2x\,dx$

34. $\int 3\sec^2 3x\,dx$

35. $\int \frac{e^x}{e^x + 3}\,dx$

36. $\int \frac{\cos x}{\sin x}\,dx$

37. $\int \frac{e^x + 3}{e^x}\,dx$

38. $\int \frac{(e^x)^2 - 2}{e^{2x}}\,dx$

39. $\int x^{1/4}(x^{5/4} - 4)\,dx$

40. $\int x^{2/3}(x^{-4/3} - 3)\,dx$

In exercises 41–52, 6 of the 12 antiderivatives can be determined using basic algebra and the derivative formulas we have presented. Find the antiderivatives of those 6 and label the others "N/A."

41. $\int \sqrt{x^3 + 4}\,dx$

42. $\int (\sqrt{x^3} + 4)\,dx$

43. $\int \frac{3x^2 - 4}{x^2}\,dx$

44. $\int \frac{x^2}{3x^2 - 4}\,dx$

45. $\int 2\sec x\,dx$

46. $\int \sec^2 x\,dx$

47. $\int 2\sin 4x\,dx$

48. $\int 2\sin^4 x\,dx$

49. $\int e^{x^2}\,dx$

50. $\int (e^x)^2\,dx$

51. $\int \left(\frac{1}{x^2} - 1\right)dx$

52. $\int \frac{1}{x^2 - 1}\,dx$

53. In example 1.12, use your CAS to evaluate the antiderivatives in parts (b) and (f). Verify that these are correct by computing the derivatives.

54. For each of the six problems in exercises 41–52 that you labeled N/A, try to find an antiderivative on your CAS. Where possible, verify that the antiderivative is correct by computing the derivatives.

In exercises 55–60, find the function $f(x)$ satisfying the given conditions.

55. $f'(x) = 4x^2 - 1, f(0) = 2$

56. $f'(x) = 4\cos x, f(0) = 3$

57. $f'(x) = 3e^x + x, f(0) = 4$

58. $f'(x) = 3\sin 2x, f(0) = 1$

59. $f''(x) = 12, f'(0) = 2, f(0) = 3$

60. $f''(x) = 2x, f'(0) = -3, f(0) = 2$

In exercises 61–64, find all functions satisfying the given conditions.

61. $f''(x) = 3\sin x + 4x^2$

62. $f''(x) = \sqrt{x} - 2\cos x$

63. $f'''(x) = 4 - 2/x^3$

64. $f'''(x) = \sin 2x - e^x$

65. Determine the position function if the velocity function is $v(t) = 3 - 12t$ and the initial position is $s(0) = 3$.

66. Determine the position function if the velocity function is $v(t) = 3e^{-t} - 2$ and the initial position is $s(0) = 0$.

67. Determine the position function if the acceleration function is $a(t) = 3\sin t + 1$, the initial velocity is $v(0) = 0$ and the initial position is $s(0) = 4$.

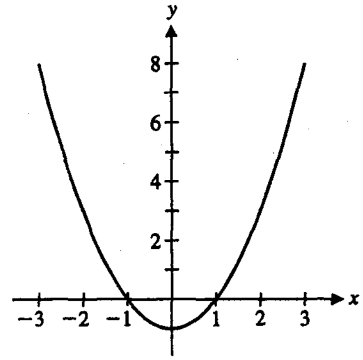
68. Determine the position function if the acceleration function is $a(t) = t^2 + 1$, the initial velocity is $v(0) = 4$ and the initial position is $s(0) = 0$.

69. Suppose that a car can accelerate from 30 mph to 50 mph in 4 seconds. Assuming a constant acceleration, find the acceleration (in miles per second squared) of the car and find the distance traveled by the car during the 4 seconds.

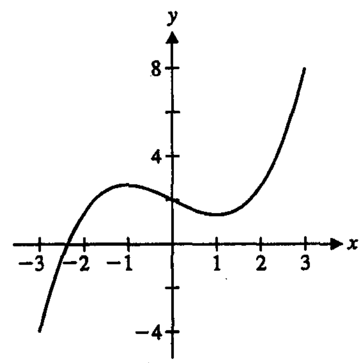
70. Suppose that a car can come to rest from 60 mph in 3 seconds. Assuming a constant (negative) acceleration, find the acceleration (in miles per second squared) of the car and find the distance traveled by the car during the 3 seconds (i.e., the stopping distance).

In exercises 71 and 72, sketch the graph of a function $f(x)$ corresponding to the given graph of $y = f'(x)$.

71.



72.



73. Sketch the graphs of three functions each of which has the derivative sketched in exercise 71.

74. Repeat exercise 71 if the given graph is of $f''(x)$.

75. For the shuttle in example 1.13, find the time when it reaches the ground and its velocity at that time. Why did we say that this would be a disastrous outcome?

76. Derive the formulas $\int \sec^2 x\, dx = \tan x + c$ and $\int \sec x \tan x\, dx = \sec x + c$.

77. Derive the formulas $\int e^x\, dx = e^x + c$ and $\int e^{-x}\, dx = -e^{-x} + c$.

78. Compute the derivatives of $e^{\sin x}$ and e^{x^2}. Given these derivatives, evaluate the indefinite integrals $\int \cos x\, e^{\sin x}\, dx$ and $\int 2x\, e^{x^2} dx$. Next, evaluate $\int x\, e^{x^2} dx$. (Hint: $\int x\, e^{x^2} dx = \frac{1}{2}\int 2x\, e^{x^2} dx$.) Similarly, evaluate $\int x^2 e^{x^3} dx$. In general, evaluate

$$\int f'(x)\, e^{f(x)}\, dx.$$

Next, evaluate $\int e^x \cos(e^x)\, dx$, $\int 2x \cos(x^2)\, dx$ and the more general

$$\int f'(x) \cos(f(x))\, dx.$$

As we have stated, there is no general rule for the antiderivative of a product, $\int f(x)g(x)\,dx$. Instead, there are many special cases that you evaluate case by case.

79. A **differential equation** is an equation involving an unknown function and one or more of its derivatives, for instance, $v'(t) = 2t + 3$. To solve this differential equation, you simply find the antiderivative $v(t) = \int (2t+3)\,dt = t^2 + 3t + c$. Notice that solutions of a differential equation are functions. In general, differential equations can be challenging to solve. For example, we introduced the differential equation $mv'(t) = -mg + kv^2(t)$ for the vertical motion of a space shuttle subject to gravity and air drag. Taking specific values of m and k gives the equation $v'(t) = -32 + 0.0003v^2(t)$. To solve this, we would need to find a function whose derivative equals -32 plus 0.0003 times the square of the function. It is difficult to find a function whose derivative is written in terms of $[v(t)]^2$ when $v(t)$ is precisely what is unknown. We can nonetheless construct a graphical representation of the solution using what is called a **direction field.** Suppose we want to construct a solution passing through the point $(0, -100)$, corresponding to an initial velocity of $v(0) = -100$ ft/s. At $t = 0$, with $v = -100$, we know that the slope of the solution is $v' = -32 + 0.0003(-100)^2 = -29$. Starting at $(0, -100)$, sketch in a short line segment with slope -29. Such a line segment would connect to the point $(1, -129)$ if you extended it that far (but make yours much shorter). At $t = 1$ and $v = -129$, the slope of the solution is $v' = -32 + 0.0003(-129)^2 \approx -27$. Sketch in a short line segment with slope -27 starting at the point $(1, -129)$. This line segment points to $(2, -156)$. At this point, $v' = -32 + 0.0003(-156)^2 \approx -24.7$. Sketch in a short line segment with slope -24.7 at $(2, -156)$. Do you see a graphical solution starting to emerge? Is the solution increasing or decreasing? concave up or concave down? If your CAS has a direction field capability, sketch the direction field and try to visualize the solution starting at points $(0, -100)$, $(0, 0)$ and $(0, -300)$.

4.2 SUMS AND SIGMA NOTATION

In section 4.1, we discussed how to calculate backward from a function describing the acceleration of an object to arrive at the function giving the position of the object at any time, t. We would now like to investigate the same process graphically. In this section, we develop an important skill necessary for this new interpretation.

Suppose that you are cruising on a highway at 60 mph. In 2 hours, you will have traveled 120 miles; in 4 hours, you will have traveled 240 miles. There's no surprise here, but notice that you can see this graphically by looking at several graphs of the (constant) velocity function $v(t) = 60$. In Figure 4.1, we have shaded in the area under the graph from $t = 0$ to $t = 2$. Notice that the area of this region equals 120, the distance covered in the time span from $t = 0$ to $t = 2$. In Figure 4.2, the shaded region from $t = 0$ to $t = 4$ has area equal to the distance of 240 miles.

So, it appears that the distance traveled over a particular time interval equals the area of the region bounded by $y = v(t)$ and the t-axis on the indicated time interval. For the

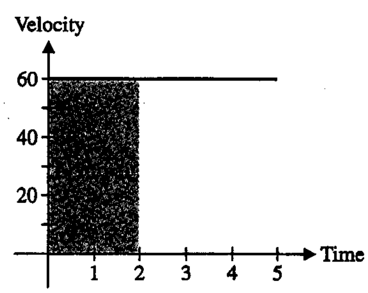


Figure 4.1
$y = v(t)$ on $[0, 2]$.

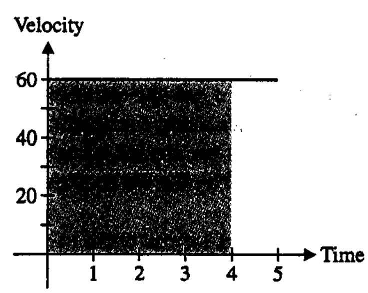


Figure 4.2
$y = v(t)$ on $[0, 4]$.