In this case, we have by the induction assumption that for $n = k + 1$,

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + \sum_{i=k+1}^{k+1} i^2 \qquad \text{Split off the last term.}$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{From (2.3).}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \qquad \text{Add the fractions.}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \qquad \text{Factor out } (k+1).$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6} \qquad \text{Combine remaining terms.}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \qquad \text{Factor the quadratic.}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \qquad \text{Rewrite the terms.}$$

$$= \frac{n(n+1)(2n+1)}{6}, \qquad \text{Since } n = k+1.$$

as desired. ■

EXERCISES

1. In the text, we mentioned that one of the benefits of using the summation notation is the simplification of calculations. To help understand this, write out in words what is meant by $\sum_{i=1}^{40}(2i^2 - 4i + 11)$.

2. Following up on exercise 1, calculate the sum $\sum_{i=1}^{40}(2i^2 - 4i + 11)$ and then describe in words how you did so. Be sure to describe any formulas and your use of them in words.

In exercises 3–6, a calculation is described in words. Translate each into summation notation and then compute the sum.

3. The sum of the squares of the first 50 positive integers.

4. The square of the sum of the first 50 positive integers.

5. The sum of the square roots of the first 10 positive integers.

6. The square root of the sum of the first 10 positive integers.

In exercises 7–14, write out all terms and compute the sums.

7. $\sum_{i=1}^{6} 3i^2$

8. $\sum_{i=1}^{5}(2i - 1)$

9. $\sum_{i=3}^{7}(i^2 + i)$

10. $\sum_{i=6}^{10}(4i + 2)$

11. $\sum_{i=0}^{7}(3i - 1)$

12. $\sum_{i=-1}^{5} 2i^3$

13. $\sum_{i=2}^{5}(\sqrt{i} + i)$

14. $\sum_{i=6}^{6}(i^2 + 2)$

In exercises 15–26, use summation rules to compute the sum.

15. $\sum_{i=1}^{70}(3i - 1)$

16. $\sum_{i=1}^{45}(3i - 4)$

17. $\sum_{i=1}^{40}(4 - i^2)$

18. $\sum_{i=1}^{50}(8 - i)$

19. $\sum_{i=1}^{100}(i^2 - 3i + 2)$

20. $\sum_{i=1}^{140}(i^2 + 2i - 4)$

21. $\sum_{i=1}^{200}(4 - 3i - i^2)$

22. $\sum_{i=1}^{250}(i^2 + 8)$

23. $\sum_{i=1}^{n}(i^2 - 3)$

24. $\sum_{i=1}^{n}(i^2 + 5)$

25. $\sum_{i=1}^{n}(4i^2 - i)$

26. $\sum_{i=1}^{n}(i^2 + 4i)$

In exercises 27–30, compute the sum and the limit of the sum as $n \to \infty$.

27. $\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right)\right]$

28. $\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right]$

29. $\sum_{i=1}^{n}\frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$

30. $\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$

In exercises 31–34, compute sums of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ for the given function and x-values, with Δx equal to the difference in adjacent x's.

31. $f(x) = x^2 + 4x$; $x = 0.2, 0.4, 0.6, 0.8, 1.0$

32. $f(x) = 3x + 5$; $x = 0.4, 0.8, 1.2, 1.6, 2.0$

33. $f(x) = 4x^2 - 2$; $x = 2.1, 2.2, 2.3, 2.4, \ldots, 3.0$

34. $f(x) = x^3 + 4$; $x = 2.05, 2.15, 2.25, 2.35, \ldots, 2.95$

35. Suppose that a car has velocity 50 mph for 2 hours, velocity 60 mph for 1 hour, velocity 70 mph for 30 minutes and velocity 60 mph for 3 hours. Find the distance traveled.

36. Suppose that a car has velocity 50 mph for 1 hour, velocity 40 mph for 1 hour, velocity 60 mph for 30 minutes and velocity 55 mph for 3 hours. Find the distance traveled.

37. Suppose that a runner has velocity 15 mph for 20 minutes, velocity 18 mph for 30 minutes, velocity 16 mph for 10 minutes and velocity 12 mph for 40 minutes. Find the distance run.

38. Suppose that a runner has velocity 12 mph for 20 minutes, velocity 14 mph for 30 minutes, velocity 18 mph for 10 minutes and velocity 15 mph for 40 minutes. Find the distance run.

39. Use mathematical induction to prove that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$.

40. Use mathematical induction to prove that $\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$ for all integers $n \geq 1$.

In exercises 41–44, use the formulas in exercises 39 and 40 to compute the sums.

41. $\sum_{i=1}^{10}(i^3 - 3i + 1)$

42. $\sum_{i=1}^{20}(i^3 + 2i)$

43. $\sum_{i=1}^{100}(i^5 - 2i^2)$

44. $\sum_{i=1}^{100}(2i^5 + 2i + 1)$

45. Prove Theorem 2.2.

46. Suppose that the velocity of a car is given by $v(t) = 3\sqrt{t} + 30$ mph at time t hours $(0 \leq t \leq 4)$. We will try to determine the distance traveled in the 4 hours. To start, we can note that the velocity at $t = 0$ is $v(0) = 3\sqrt{0} + 30 = 30$ mph and the velocity at time $t = 1$ is $v(1) = 3\sqrt{1} + 30 = 33$ mph. Since the average of these velocities is 31.5 mph, we could estimate that the car traveled 31.5 miles in the first hour. Carefully explain why this is not necessarily correct. Even so, it will serve as a first approximation. Since $v(1) = 33$ mph and $v(2) = 3\sqrt{2} + 30 \approx 34$ mph, we can estimate that the car traveled 33.5 mph in the second hour. Using $v(3) \approx 35$ mph and $v(4) = 36$ mph, find similar estimates for the distance traveled in the third and fourth hours, and then estimate the total distance. To improve this estimate, we can find an estimate for the distance covered each half-hour. The first estimate would take $v(0) = 30$ mph and $v(0.5) \approx 32.1$ mph and estimate an average velocity of 31.05 mph and a distance of 15.525 miles. Estimate the average velocity and then the distance for the remaining 7 half-hours and estimate the total distance. We can improve this estimate, too. By estimating the average velocity every quarter-hour, find a third estimate of the total distance. Based on these three estimates, conjecture the limit of these approximations as the time interval considered goes to zero.

47. In this exercise, we investigate a generalization of sums called an **infinite series.** Suppose a bouncing ball has **coefficient of restitution** equal to 0.6. This means that if the ball hits the ground with velocity v ft/s, it rebounds with velocity $0.6v$. Ignoring air resistance, a ball launched with velocity v ft/s will stay in the air $v/16$ seconds before hitting the ground. Suppose a ball with coefficient of restitution 0.6 is launched with initial velocity 60 ft/s. Explain why the total time in the air is given by $60/16 + (0.6)(60)/16 + (0.6)(0.6)(60)/16 + \cdots$. It might seem like the ball would continue to bounce forever. To see otherwise, compute sums using more and more terms following this pattern. Find the limit that these sums approach. The limit is the number of seconds that the ball continues to bounce.