on the interval [0, 1], using $n = 10$. You should note that in this case (as with any increasing function), the rectangles corresponding to the right endpoint evaluation (Figure 4.13a) give too much area on each subinterval, while the rectangles corresponding to left endpoint evaluation (Figure 4.13b) give too little area. We leave it to you to observe that the reverse is true for a decreasing function.

Example 3.5 **Computing Riemann Sums with Different Evaluation Points**

Compute Riemann sums for $f(x) = \sqrt{x+1}$ on the interval [1, 3], for $n = 10, 50, 100, 500, 1000$ and 5000, using the left endpoint, right endpoint and midpoint of each subinterval as the evaluation points.

Solution The numbers given in the following table are from a program written for a programmable calculator. We suggest that you test your own program or one built into your CAS against these values (rounded off to six digits).

n	Left Endpoint	Midpoint	Right Endpoint
10	3.38879	3.44789	3.50595
50	3.43598	3.44772	3.45942
100	3.44185	3.44772	3.45357
500	3.44654	3.44772	3.44888
1000	3.44713	3.44772	3.44830
5000	3.44760	3.44772	3.44783

There are several conclusions to be drawn from these numbers. First, there is good evidence that all three sets of numbers are converging to a common limit of approximately 3.4477. You should notice that the limit is independent of the particular evaluation point used. Second, even though the limits are the same, the different rules approach the limit at different rates. You should try computing left and right endpoint sums for larger values of n, to see that these eventually approach 3.44772, also. ■

Riemann sums using midpoint evaluation usually approach the limit far faster than left or right endpoint rules. If you think about the rectangles being drawn, you may be able to explain why. Finally, notice that the left and right endpoint sums in example 3.5 approach the limit from opposite directions and at about the same rate. We take advantage of this in an approximation technique called the Trapezoidal Rule, to be discussed in section 4.7. If your CAS or graphics calculator does not have a command for calculating Riemann sums, we suggest that you write a program for computing them yourself.

EXERCISES

1. It turns out that for many functions, the limit of the Riemann sums is independent of the choice of evaluation points. Discuss why this is a somewhat surprising result. To make the result more believable, consider a continuous function $f(x)$. As the number of partition points gets larger, the distance between the endpoints gets smaller. For the continuous function $f(x)$, explain why the difference between the function values at any two points in a given subinterval will have to get smaller.

2. Rectangles are not the only basic geometric shapes for which we have an area formula. Discuss how you might approximate the area under a parabola using circles or triangles. Which geometric shape do you think is the easiest to use?

In exercises 3–10, list the evaluation points corresponding to the midpoint of each subinterval, sketch the function and approximating rectangles and evaluate the Riemann sum.

3. $f(x) = x^2 + 1, [0, 1], n = 4$

4. $f(x) = x^2 + 1, [0, 2], n = 4$

5. $f(x) = x^3 - 1, [1, 2], n = 4$

6. $f(x) = x^3 - 1, [1, 3], n = 4$

7. $f(x) = \sin x, [0, \pi], n = 4$

8. $f(x) = \sin x, [0, \pi], n = 8$

9. $f(x) = 4 - x^2, [-1, 1], n = 4$

10. $f(x) = 4 - x^2, [-3, -1], n = 4$

In exercises 11–26, approximate the area under the curve on the given interval using n rectangles and the indicated evaluation rule.

11. $y = x^2$ on $[0, 1]$, $n = 8$, midpoint evaluation

12. $y = x^2$ on $[0, 1]$, $n = 8$, right-endpoint evaluation

13. $y = x^2$ on $[-1, 1]$, $n = 8$, left-endpoint evaluation

14. $y = x^2$ on $[-1, 1]$, $n = 8$, midpoint evaluation

15. $y = \sqrt{x+2}$ on $[1, 4]$, $n = 8$, midpoint evaluation

16. $y = \sqrt{x+2}$ on $[1, 4]$, $n = 8$, right-endpoint evaluation

17. $y = e^{-2x}$ on $[-1, 1]$, $n = 16$, left-endpoint evaluation

18. $y = e^{-2x}$ on $[-1, 1]$, $n = 16$, midpoint evaluation

19. $y = \cos x$ on $[0, \pi/2]$, $n = 50$, midpoint evaluation

20. $y = \cos x$ on $[0, \pi/2]$, $n = 100$, right-endpoint evaluation

21. $y = 3x - 2$ on $[1, 4]$, $n = 4$, midpoint evaluation

22. $y = 3x - 2$ on $[1, 4]$, $n = 40$, midpoint evaluation

23. $y = x^3 - 1$ on $[1, 3]$, $n = 100$, midpoint evaluation

24. $y = x^3 - 1$ on $[1, 3]$, $n = 100$, right-endpoint evaluation

25. $y = x^3 - 1$ on $[-1, 1]$, $n = 100$, left-endpoint evaluation

26. $y = x^3 - 1$ on $[-1, 1]$, $n = 100$, right-endpoint evaluation

In exercises 27–30, construct a table of Riemann sums as in Example 3.5 to show that sums with right-endpoint, midpoint and left-endpoint evaluation all converge to the same value as $n \to \infty$.

27. $f(x) = 4 - x^2, [-2, 2]$

28. $f(x) = \sin x, [0, \pi/2]$

29. $f(x) = x^3 - 1, [1, 3]$

30. $f(x) = x^3 - 1, [-1, 1]$

In exercises 31–34, use Riemann sums and a limit to compute the exact area under the curve.

31. $y = x^2 + 2$ on $[0, 1]$

32. $y = x^2 + 3x$ on $[0, 1]$

33. $y = 2x^2 + 1$ on $[1, 3]$

34. $y = 4x + 2$ on $[1, 3]$

In exercises 35–40, use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation.

35.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	2.0	2.4	2.6	2.7	2.6	2.4	2.0	1.4	0.6

36.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	3.0	2.2	1.6	0.7	0.6	0.4	−0.2	0.4	0.6

37.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	1.0	1.4	2.1	2.7	2.6	2.8	3.0	3.4	3.6

38.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	2.0	2.2	1.6	1.4	1.6	2.0	2.2	2.4	2.0

39.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.8	1.4	1.1	0.7	1.2	1.4	1.8	2.4	2.6

40.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6
$f(x)$	0.0	0.4	0.6	0.8	1.2	1.4	1.2	1.4	1.0

In exercises 41–44, graphically determine whether a Riemann sum with (a) left-endpoint, (b) midpoint and (c) right-endpoint evaluation points will be greater than or less than the area under the curve $y = f(x)$ on $[a, b]$.

41. $f(x)$ is increasing and concave up on $[a, b]$.

42. $f(x)$ is increasing and concave down on $[a, b]$.

43. $f(x)$ is decreasing and concave up on $[a, b]$.

44. $f(x)$ is decreasing and concave down on $[a, b]$.

45. For the function $f(x) = x^2$ on the interval $[0, 1]$, by trial and error find evaluation points for $n = 2$ such that the Riemann sum equals the exact area of $1/3$.

46. For the function $f(x) = \sqrt{x}$ on the interval $[0, 1]$, by trial and error find evaluation points for $n = 2$ such that the Riemann sum equals the exact area of $2/3$.

47. Show that for right-endpoint evaluation on the interval $[a, b]$ with each subinterval of length $\Delta x = (b - a)/n$, the evaluation points are $c_i = a + i\Delta x$, for $i = 1, 2, \ldots, n$.

48. Show that for left-endpoint evaluation on the interval $[a, b]$ with each subinterval of length $\Delta x = (b - a)/n$, the evaluation points are $c_i = a + (i - 1)\Delta x$, for $i = 1, 2, \ldots, n$.

49. As in exercises 47 and 48, find a formula for the evaluation points for midpoint evaluation.

50. As in exercises 47 and 48, find a formula for evaluation points that are one-third of the way from the left-endpoint to the right-endpoint.

51. Riemann sums can also be defined on **irregular partitions** for which subintervals are not of equal size. An example of an irregular partition of the interval $[0, 1]$ is $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 0.9$, $x_4 = 1$. Explain why the corresponding Riemann sum would be

$$f(c_1)(0.2) + f(c_2)(0.4) + f(c_3)(0.3) + f(c_4)(0.1),$$

for evaluation points c_1, c_2, c_3 and c_4. Identify the interval from which each c_i must be chosen and give examples of evaluation points. To see why irregular partitions might be useful, consider the function $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 1 & \text{if } x \geq 1 \end{cases}$ on the interval $[0, 2]$. One way to approximate the area under the graph of this function is to compute Riemann sums using midpoint evaluation for $n = 10$, $n = 50$, $n = 100$ and so on. Show graphically and numerically that with midpoint evaluation, the Riemann sum with $n = 2$ gives the correct area on the subinterval $[0, 1]$. Then explain why it would be wasteful to compute Riemann sums on this subinterval for larger and larger values of n. A more efficient strategy would be to compute the areas on $[0, 1]$ and $[1, 2]$ separately and add them together. The area on $[0, 1]$ can be computed exactly using a small value of n, while the area on $[1, 2]$ must be approximated using larger and larger values of n. Use this technique to estimate the area for $f(x)$ on the interval $[0, 2]$. Try to determine the area to within an error of 0.01 (discuss why you believe your answer is this accurate).

52. Graph the function $f(x) = e^{-x^2}$. You may recognize this curve as the so-called "bell curve," which is of fundamental importance in statistics. We define the **area function** $g(t)$ to be the area between this graph and the x-axis between $x = 0$ and $x = t$ (for now, assume that $t > 0$). Sketch the area that defines $g(1)$ and $g(2)$ and argue that $g(2) > g(1)$. Explain why the function $g(x)$ is increasing and hence $g'(x) > 0$ for $x > 0$. Further, argue that $g'(2) < g'(1)$. Explain why $g'(x)$ is a decreasing function. Thus, $g'(x)$ has the same general properties (positive, decreasing) that $f(x)$ does. In fact, we will discover in section 4.5 that $g'(x) = f(x)$. To collect some evidence for this result, use Riemann sums to estimate $g(2)$, $g(1.1)$, $g(1.01)$ and $g(1)$. Use these values to estimate $g'(1)$ and compare to $f(1)$.

4.4 THE DEFINITE INTEGRAL

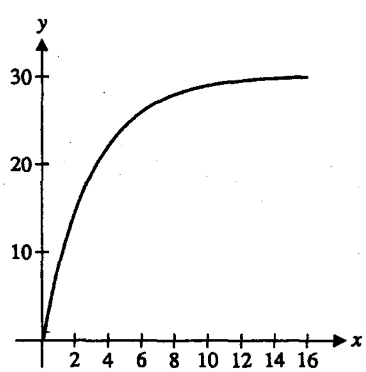

Figure 4.14a
$y = f(x)$.

Throughout this chapter, we have been considering the problem of computing distance from a given velocity function. For instance, suppose that a sky diver starts her jump by stepping out of an airplane (so that she starts with zero downward velocity). The jumper gradually picks up speed until a maximum speed is reached. Such a *terminal velocity* is the speed at which the force due to air resistance cancels out the force due to gravity. A function that has these properties is $f(x) = 30(1 - e^{-x/3})$ (see Figure 4.14a), which we can think of as modeling the (downward) velocity x seconds into the jump.

As we discussed in section 4.2, the area under this curve on the interval $0 \leq x \leq t$ corresponds to the distance fallen in the first t seconds. On a given interval (i.e., for a given value of t), we can approximate this area by first partitioning the interval into n subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, of equal width, Δx. On each subinterval, we construct a rectangle of height $f(c_i)$, for any choice of $c_i \in [x_{i-1}, x_i]$ (see Figure 4.14b). Finally, summing the areas of the rectangles gives us an approximation to the area,

$$A \approx \sum_{i=1}^{n} f(c_i)\,\Delta x.$$