$x \in [a, b]$, then inequality (4.4) holds:

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Along with its role in the proof of the Integral Mean Value Theorem, this has some additional significance, in that it enables us to estimate the value of a definite integral. Although the estimate is generally only a rough one, it still has importance in that it gives us an interval in which the value must lie. We illustrate this in the following example.

Example 4.8 **Estimating the Value of an Integral**

Use inequality (4.4) to estimate the value of $\int_0^1 \sqrt{x^2+1}\,dx$.

Solution First, notice that it's beyond your present abilities to compute the value of this integral exactly. However, notice that

$$1 \le \sqrt{x^2+1} \le \sqrt{2}, \text{ for all } x \in [0, 1].$$

From inequality (4.4), we now have

$$1 \le \int_0^1 \sqrt{x^2+1}\,dx \le \sqrt{2} \approx 1.414214.$$

In other words, although we still do not know the exact value of the integral, we know that it must be between 1 and $\sqrt{2} \approx 1.414214$.

EXERCISES

1. Sketch a graph of a function f that is both positive and negative on an interval $[a, b]$. Explain in terms of area what it means to have $\int_a^b f(x)\,dx = 0$. Also, explain what it means to have $\int_a^b f(x)\,dx \ge 0$ and $\int_a^b f(x)\,dx \le 0$.

2. To get a physical interpretation of the result in Theorem 4.3, suppose that $f(x)$ and $g(x)$ are velocity functions for two different objects starting at the same position. If $f(x) \ge g(x) \ge 0$, explain why it follows that $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

3. The Integral Mean Value Theorem says that if $f(x)$ is continuous on the interval $[a, b]$, then there exists a number c between a and b such that $f(c)(b-a) = \int_a^b f(x)\,dx$. By thinking of the left-hand side of this equation as the area of a rectangle, sketch a picture that illustrates this result, and explain why the result follows.

4. Write out the Integral Mean Value Theorem as applied to the derivative $f'(x)$. Then write out the Mean Value Theorem for derivatives (see section 2.9). If the c-values identified by each theorem are the same, what does $\int_a^b f'(x)\,dx$ have to equal? Explain why, at this point, we don't know whether or not the c-values are the same.

In exercises 5–10, use Riemann sums to estimate the value of the integral (obtain two digits of accuracy).

5. $\int_0^3 (x^3 + x)\,dx$

6. $\int_1^2 (x^2 - 1)\,dx$

7. $\int_2^4 \frac{1}{x^2}\,dx$

8. $\int_0^3 \sqrt{x^2+1}\,dx$

9. $\int_0^\pi \sin x^2\,dx$

10. $\int_{-2}^2 e^{-x^2}\,dx$

In exercises 11–16, evaluate the integral by computing the limit of Riemann sums.

11. $\int_0^1 2x\,dx$

12. $\int_1^2 2x\,dx$

13. $\int_0^2 x^2\,dx$

14. $\int_0^3 (x^2+1)\,dx$

15. $\int_1^3 (2x-1)\,dx$

16. $\int_{-2}^2 (x^2-1)\,dx$

In exercises 17–26, write the given (total) area as an integral or sum of integrals.

17. The area above the x-axis and below $y = 4 - x^2$

18. The area above the x-axis and below $y = 4x - x^2$

19. The area below the x-axis and above $y = x^2 - 4$

20. The area below the x-axis and above $y = x^2 - 4x$

21. The area of the region bounded by $y = x^2$, $x = 2$ and the x-axis

22. The area of the region bounded by $y = x^3$, $x = 3$ and the x-axis

23. The area between $y = \sin x$ and the x-axis for $0 \le x \le \pi$

24. The area between $y = \sin x$ and the x-axis for $-\pi/2 \le x \le \pi/4$

25. The area between $y = x^3 - 3x^2 + 2x$ and the x-axis for $0 \le x \le 2$

26. The area between $y = x^3 - 4x$ and the x-axis for $-2 \le x \le 3$

In exercises 27–30, use the given velocity function and initial position to estimate the final position $s(b)$.

27. $v(t) = 60 - 16t$, $s(0) = 2$, $b = 2$

28. $v(t) = 20 + 10t$, $s(1) = 3$, $b = 3$

29. $v(t) = 40(1 - e^{-2t})$, $s(0) = 0$, $b = 4$

30. $v(t) = 30e^{-t/4}$, $s(0) = -1$, $b = 4$

In exercises 31–34, use Theorem 4.2 to write the expression as a single integral.

31. $\int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$

32. $\int_0^3 f(x)\,dx - \int_2^3 f(x)\,dx$

33. $\int_0^2 f(x)\,dx + \int_2^1 f(x)\,dx$

34. $\int_{-1}^2 f(x)\,dx + \int_2^3 f(x)\,dx$

In exercises 35–38, sketch the area corresponding to the integral.

35. $\int_1^2 (x^2 - x)\,dx$

36. $\int_2^4 (x^2 - x)\,dx$

37. $\int_0^{\pi/2} \cos x\,dx$

38. $\int_{-2}^2 e^{-x}\,dx$

In exercises 39–42, use the Integral Mean Value Theorem to estimate the value of the integral.

39. $\int_{\pi/3}^{\pi/2} 3\cos x^2\,dx$

40. $\int_0^{1/2} e^{-x^2}\,dx$

41. $\int_0^2 \sqrt{2x^2+1}\,dx$

42. $\int_{-1}^1 \frac{3}{x^3+2}\,dx$

In exercises 43 and 44, find a value of c that satisfies the conclusion of the Integral Mean Value Theorem.

43. $\int_0^2 3x^2\,dx\ (= 8)$

44. $\int_{-1}^1 (x^2 - 2x)\,dx\ (= \frac{2}{3})$

45. Substitute $f(x) = g'(x)$ in the conclusion of the Integral Mean Value Theorem. Discuss how the result compares with the Mean Value Theorem of section 2.9.

46. Prove that if f is continuous on the interval $[a, b]$, then there exists a number c in (a, b) such that $f(c)$ equals the average value of f on the interval $[a, b]$.

47. Prove part (iv) of Theorem 4.2 for the special case $c = \frac{1}{2}(a+b)$.

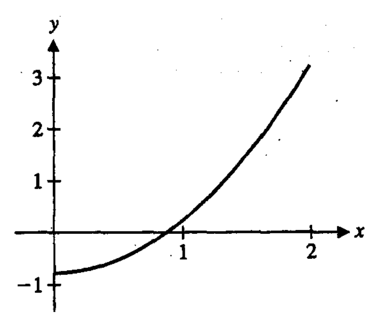
48. Show that parts (i)–(iii) of Theorem 4.2 must follow if it is true that

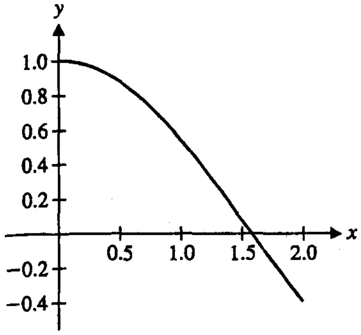
$$\int_a^b [cf(x) + dg(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx.$$

(Hint: Make clever choices of c and d, like $d = 0$ or $c = d = 1$.)

In exercises 49–52, use the graph to determine whether $\int_0^2 f(x)dx$ is positive or negative.

49.



50.


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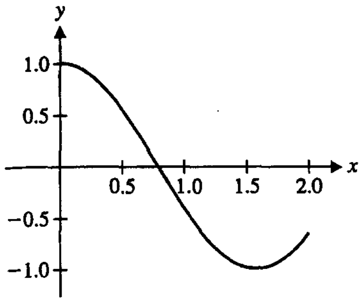

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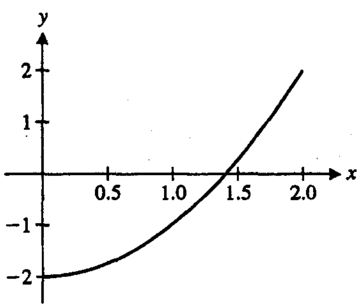

53. For the functions $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2+2 & \text{if } x \geq 1 \end{cases}$ and $g(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ x^2+2 & \text{if } x > 1 \end{cases}$, assume that $\int_0^2 f(x)\,dx$ and $\int_0^2 g(x)\,dx$ exist. Explain why the approximating Riemann sums with midpoint evaluations are equal for any even value of n. Argue that this result implies that the two integrals are equal.

54. For the functions defined in exercise 53, explain why the integrals are both equal to the sum $\int_0^1 2x\,dx + \int_1^2 (x^2+2)\,dx$.

In exercises 55–58, compute $\int_0^4 f(x)dx$.

55. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 & \text{if } x \geq 1 \end{cases}$

56. $f(x) = \begin{cases} 2 & \text{if } x < 2 \\ 3x & \text{if } x \geq 2 \end{cases}$

57. $f(x) = \begin{cases} 4 & \text{if } x < 3 \\ x+1 & \text{if } x \geq 3 \end{cases}$

58. $f(x) = \begin{cases} x-1 & \text{if } x < 1 \\ 2 & \text{if } x \geq 1 \end{cases}$

59. Suppose that, for a particular population of organisms, the birth rate is given by $b(t) = 410 - 0.3t$ organisms per month and the death rate is given by $a(t) = 390 + 0.2t$ organisms per month. Explain why $\int_0^{12} [b(t) - a(t)]\,dt$ represents the net change in population in the first 12 months. Determine for which values of t it is true that $b(t) > a(t)$. At which times is the population increasing? decreasing? Determine the time at which the population reaches a maximum.

60. Suppose that, for a particular population of organisms, the birth rate is given by $b(t) = 400 - 3\sin t$ organisms per month and the death rate is given by $a(t) = 390 + 0.2t$ organisms per month. Explain why $\int_0^{12} [b(t) - a(t)]\,dt$ represents the net change in population in the first 12 months. Graphically determine for which values of t it is true that $b(t) > a(t)$. At which times is the population increasing? decreasing? Estimate the time at which the population reaches a maximum.

61. For a particular ideal gas at constant temperature, pressure P and volume V are related by $PV = 10$. The work required to increase the volume from $V = 2$ to $V = 4$ is given by the integral $\int_2^4 P(V)\,dV$. Estimate the value of this integral.

62. Suppose that the temperature t months into the year is given by $T(t) = 64 - 24\cos\frac{\pi}{6}t$ (degrees Fahrenheit). Estimate the average temperature over an entire year. Explain why this answer is obvious from the graph of $T(t)$.

In exercises 63–68, estimate the average value of the function on the given interval.

63. $f(x) = 2x + 1$, $[0, 4]$

64. $f(x) = x^2 + 2x$, $[0, 1]$

65. $f(x) = x^2 - 1$, $[1, 3]$

66. $f(x) = 2x - 2x^2$, $[0, 1]$

67. $f(x) = \cos x$, $[0, \pi/2]$

68. $f(x) = \sin x$, $[0, \pi/2]$

69. The *impulse-momentum equation* states the relationship between a force $F(t)$ applied to an object of mass m and the resulting change in velocity Δv of the object. The equation is $m\Delta v = \int_a^b F(t)\,dt$, where $\Delta v = v(b) - v(a)$. Suppose that the force of a baseball bat on a ball is approximately $F(t) = 9 - 10^8(t - 0.0003)^2$ thousand pounds for t between 0 and 0.0006 seconds. What is the maximum force on the ball? Using $m = 0.01$ for the mass of a baseball, estimate the change in velocity Δv (in ft/s).

70. Measurements taken of the feet of badminton players lunging for a shot indicate a vertical force of approximately $F(t) = 1000 - 25{,}000(t - 0.2)^2$ Newtons for t between 0 and 0.4 seconds (see *The Science of Racquet Sports*). For a player of mass

$m = 5$, use the impulse-momentum equation in exercise 69 to estimate the change in vertical velocity of the player.

71. Use the Integral Mean Value Theorem to prove the following fact for a continuous function. If the evaluation point is chosen properly, the Riemann sum approximation of $\int_a^b f(x)\,dx$ with $n = 1$ can be made to be exact.

72. Use the Integral Mean Value Theorem to prove the following fact for a continuous function. For any positive integer n, there exists a set of evaluation points for which the Riemann sum approximation of $\int_a^b f(x)\,dx$ is exact.

73. Many of the basic quantities used by epidemiologists to study the spread of disease are described by integrals. In the case of AIDS, a person becomes infected with the HIV virus and, after an incubation period, develops AIDS. Our goal is to derive a formula for the number of AIDS cases given the HIV infection rate $g(t)$ and the incubation distribution $F(t)$. To take a simple case, suppose that the infection rate the first month is 20 people per month, the infection rate the second month is 30 people per month and the infection rate the third month is 25 people per month. Then $g(1) = 20$, $g(2) = 30$ and $g(3) = 25$. Also, suppose that 20% of those infected develop AIDS after 1 month, 50% develop AIDS after 2 months and 30% develop AIDS after 3 months (fortunately, these figures are not at all realistic). Then $F(1) = 0.2$, $F(2) = 0.5$ and $F(3) = 0.3$. Explain why the number of people developing AIDS in the fourth month would be $g(1)F(3) + g(2)F(2) + g(3)F(1)$. Compute this number. Next, suppose that $g(0.5) = 16$, $g(1) = 20$, $g(1.5) = 26$, $g(2) = 30$, $g(2.5) = 28$, $g(3) = 25$ and $g(3.5) = 22$. Further, suppose that $F(0.5) = 0.1$, $F(1) = 0.1$, $F(1.5) = 0.2$, $F(2) = 0.3$, $F(2.5) = 0.1$, $F(3) = 0.1$ and $F(3.5) = 0.1$. Compute the number of people developing AIDS in the fourth month. If we have $g(t)$ and $F(t)$ defined at all real numbers t, explain why the number of people developing AIDS in the fourth month equals $\int_0^4 g(t)F(4-t)\,dt$.

4.5 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section, we present a pair of results known collectively as the Fundamental Theorem of Calculus. What could these results be to make them **fundamental** to calculus? On a practical level, the Fundamental Theorem provides us with a much-needed shortcut for computing definite integrals. Just as the power rule and other basic differentiation formulas relieved us of the burden of using the limit definition of the derivative, the Fundamental Theorem gives us a powerful tool for computing integrals symbolically without struggling to find limits of Riemann sums. Given that we can presently compute such limits exactly only for a very small number of functions, this theorem takes on even greater significance.

On a conceptual level, the Fundamental Theorem unifies the seemingly disconnected studies of derivatives and definite integrals. In the rush of learning the rules for computing derivatives and definite integrals, there is barely time to wonder why these concepts are in the same course. The Fundamental Theorem shows us that differentiation and integration are, in fact, inverse processes. In this sense, the theorem is truly **fundamental** to calculus as a coherent discipline.

We have dropped a few hints as to the nature of the first part of the Fundamental Theorem. First, recall that we used suspiciously similar notations for indefinite and definite integrals. We have also used both antidifferentiation and area calculations to compute distance from velocity. However, the Fundamental Theorem makes much stronger statements about the relationship between differentiation and integration.

NOTES

The Fundamental [illegible] says that to compute [illegible] integral, we need only [illegible] antiderivative and [illegible] at the two limits of [illegible]. Observe that this is a [illegible] improvement over [illegible] of Riemann sums, which [illegible] compute exactly for only [illegible] simple cases.

Theorem 5.1 (Fundamental [illegible])

If f is continuous on $[a, b]$ and [illegible]

$$\int_a^b f(x)\,dx = [illegible]$$