In the following example, we use Part II of the Fundamental Theorem to determine information about a seemingly very complicated function. Notice that although we don't know how to evaluate the integral, we can use the Fundamental Theorem to obtain some important information about the function.

**Example 5.12** **Finding a Tangent Line for a Function Defined as an Integral**

For the function $F(x) = \int_4^{x^2} \ln(t^3 + 4)\, dt$, find an equation of the tangent line at $x = 2$.

**Solution** Notice that there are almost no function values that we can compute exactly, yet we can easily find an equation of a tangent line! From Part II of the Fundamental Theorem and the chain rule, we get the derivative

$$F'(x) = \ln[(x^2)^3 + 4]\frac{d}{dx}(x^2) = \ln[(x^2)^3 + 4](2x).$$

So, the slope at $x = 2$ is $F'(2) = \ln(68)(4) \approx 16.878$. The tangent passes through the point $x = 2$ and $y = F(2) = \int_4^4 \ln(t^3 + 4)\, dt = 0$ (since the upper limit equals the lower limit). An equation of the tangent line is then

$$y = 4\ln 68(x - 2).$$

At the beginning of this section, we briefly discussed the theoretical importance of the Fundamental Theorem in unifying the differential and integral calculus. Both parts of the theorem involve both derivatives and integrals and this connection is certainly important. Now that we've discussed the computational significance of the results, we want to close the section by observing that the two parts of the Fundamental Theorem are different sides of the same theoretical coin. In Part I, we observe that if we can find an antiderivative for the function in the integrand, then we can easily evaluate the definite integral. In Part II, we observe that every continuous function has an antiderivative, in the form of a definite integral with a variable limit of integration.

Recall the conclusions of Parts I and II of the Fundamental Theorem:

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

and

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x).$$

In both cases, we are saying that differentiation and integration are in some sense *inverse operations*: their effects in a certain sense (with appropriate hypotheses) cancel each other out.

## EXERCISES 4.5

1. To explore Part I of the Fundamental Theorem graphically, first suppose that $F(x)$ is increasing on the interval $[a, b]$. Explain why both of the expressions $F(b) - F(a)$ and $\int_a^b F'(x)\, dx$ will be positive. Further, explain why the faster $F(x)$ increases, the larger each expression will be. Similarly, explain why if $F(x)$ is decreasing both expressions will be negative.

2. You can think of Part I of the Fundamental Theorem in terms of position $s(t)$ and velocity $v(t) = s'(t)$. Start

by assuming that $v(t) \geq 0$. Explain why $\int_a^b v(t)\,dt$ gives the total distance traveled and explain why this equals $s(b) - s(a)$. Discuss what changes if $v(t) < 0$.

3. To explore Part II of the Fundamental Theorem graphically, consider the function $g(x) = \int_a^x f(t)\,dt$. If $f(t)$ is positive on the interval $[a, b]$, explain why $g'(x)$ will also be positive. Further, the larger $f(t)$ is, the larger $g'(x)$ will be. Similarly, explain why if $f(t)$ is negative then $g'(x)$ will also be negative.

4. In Part I of the Fundamental Theorem, $F$ can be *any* antiderivative of $f$. Recall that any two antiderivatives of $f$ differ by a constant. Explain why $F(b) - F(a)$ is **well-defined;** that is, if $F_1$ and $F_2$ are different antiderivatives, explain why $F_1(b) - F_1(a) = F_2(b) - F_2(a)$. When evaluating a definite integral, explain why you do not need to include "$+c$" with the antiderivative.

**In exercises 5–30, use Part I of the Fundamental Theorem to compute each integral exactly.**

5. $\int_0^2 (2x - 3)\,dx$

6. $\int_0^3 (x^2 - 2)\,dx$

7. $\int_{-1}^1 (x^3 + 2x)\,dx$

8. $\int_0^2 (x^3 + 3x - 1)\,dx$

9. $\int_0^4 (\sqrt{x} + 3x)\,dx$

10. $\int_1^2 (4x - 2/x^2)\,dx$

11. $\int_0^1 (x\sqrt{x} + x^{-1/2})\,dx$

12. $\int_0^8 (\sqrt[3]{x} - x^{2/3})\,dx$

13. $\int_0^{\pi/2} 2\sin x\,dx$

14. $\int_{-\pi}^0 4\cos x\,dx$

15. $\int_0^{\pi/4} \sec x \tan x\,dx$

16. $\int_0^{\pi/4} \sec^2 x\,dx$

17. $\int_{\pi/2}^{\pi} (2\sin x - \cos x)\,dx$

18. $\int_0^{\pi} 3\sin 2x\,dx$

19. $\int_0^1 (e^x - e^{-x})\,dx$

20. $\int_{-1}^1 (e^x + e^{-x})\,dx$

21. $\int_0^3 (3e^{2x} - x^2)\,dx$

22. $\int_0^2 (3x - e^{-3x})\,dx$

23. $\int_{-1}^1 (e^x + e^{-x})^2\,dx$

24. $\int_0^1 \frac{e^{2x} - 1}{e^x}\,dx$

25. $\int_1^4 \frac{x - 3}{x}\,dx$

26. $\int_1^2 \frac{x^2 - 3x + 4}{x^2}\,dx$

27. $\int_0^4 x(x - 2)\,dx$

28. $\int_0^{\pi/3} \frac{3}{\cos^2 x}\,dx$

29. $\int_0^3 (x^3 - \sin x)\,dx$

30. $\int_{-2}^0 (2x + e^x)\,dx$

**In exercises 31–40, use the Fundamental Theorem if possible or estimate the integral using Riemann sums. (Hint: Six problems can be worked using antiderivative formulas we have covered so far.)**

31. $\int_0^2 \sqrt{x^2 + 1}\,dx$

32. $\int_0^2 (\sqrt{x} + 1)^2\,dx$

33. $\int_0^1 (e^x + 1)^2\,dx$

34. $\int_0^1 e^{x^2+1}\,dx$

35. $\int_1^4 \frac{x^2}{x^2 + 4}\,dx$

36. $\int_1^4 \frac{x^2 + 4}{x^2}\,dx$

37. $\int_0^{\pi} \sin x^2\,dx$

38. $\int_0^{\pi} \sin 2x\,dx$

39. $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

40. $\int_0^{\pi/4} \frac{\tan x}{\sec x}\,dx$

**In exercises 41–46, find the derivative $f'(x)$.**

41. $f(x) = \int_0^x (t^2 - 3t + 2)\,dt$

42. $f(x) = \int_2^x (t^2 - 3t - 4)\,dt$

43. $f(x) = \int_0^{x^2} \left(e^{-t^2} + 1\right)dt$

44. $f(x) = \int_2^{x^2} \left(e^{t^2} - 1\right)dt$

45. $f(x) = \int_x^{-1} \ln(t^2 + 1)\,dt$

46. $f(x) = \int_x^2 \sec t\,dt$

**In exercises 47–54, find the given area.**

47. The area above the $x$-axis and below $y = 4 - x^2$

48. The area above the $x$-axis and below $y = 4x - x^2$

49. The area below the $x$-axis and above $y = x^2 - 4$

50. The area below the $x$-axis and above $y = x^2 - 4x$

51. The area of the region bounded by $y = x^2$, $x = 2$ and the $x$-axis

52. The area of the region bounded by $y = x^3$, $x = 3$ and the $x$-axis

53. The area between $y = \sin x$ and the $x$-axis for $0 \leq x \leq \pi$

54. The area between $y = \sin x$ and the $x$-axis for $-\pi/2 \leq x \leq \pi/4$

**In exercises 55–58, find an equation of the tangent line at the given value of $x$.**

55. $y = \int_0^x \sin\sqrt{t^2+\pi^2}\,dt,\ x = 0$

56. $y = \int_{-1}^x \ln(t^2+2t+2)\,dt,\ x = -1$

57. $y = \int_2^x \cos(\pi t^3)\,dt,\ x = 2$

58. $y = \int_0^x e^{-t^2+1}\,dt,\ x = 0$

59. Use the derivative in exercise 41 to locate and identify all local extrema of $f(x) = \int_0^x (t^2 - 3t + 2)\,dt$.

60. Katie drives a car at speed $f(t) = 55 + 10\cos\frac{\pi}{2}t$ mph and Michael drives a car at speed $g(t) = 50 + 2t$ mph at time $t$ minutes. Suppose that Katie and Michael are at the same location at time $t = 0$. Compute $\int_0^x [f(t) - g(t)]\,dt$ and interpret the integral in terms of a race between Katie and Michael.

**In exercises 61 and 62, (a) explain how you know the proposed integral value is wrong and (b) find all mistakes.**

61. $\int_{-1}^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{x=-1}^{x=1} = -1 - (1) = -2$

62. $\int_0^\pi \sec^2 x\,dx = \tan x\bigg|_{x=0}^{x=\pi} = \tan\pi - \tan 0 = 0$

**In exercises 63–70, find the position function $s(t)$ from the given velocity or acceleration function and initial value(s). Assume that units are feet and seconds.**

63. $v(t) = 40 - \sin t,\ s(0) = 2$

64. $v(t) = 30 + 4\cos 3t,\ s(0) = 0$

65. $v(t) = 25(1 - e^{-2t}),\ s(0) = 0$

66. $v(t) = 10e^{-t},\ s(0) = 2$

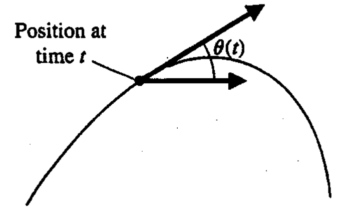
67. $a(t) = 4 - t,\ v(0) = 8,\ s(0) = 0$

68. $a(t) = 16 - t^2,\ v(0) = 0,\ s(0) = 30$

69. $a(t) = 24 + e^{-t},\ v(0) = 0,\ s(0) = 0$

70. $a(t) = 3e^{-2t},\ v(0) = -4,\ s(0) = 0$

71. If $\theta(t)$ is the angle between the path of a moving object and a fixed ray (see the figure in the next column), the **angular velocity** of the object is $\omega(t) = \theta'(t)$ and the **angular acceleration** of the object is $\alpha(t) = \omega'(t)$.




Suppose a baseball batter swings with a constant angular acceleration of $\alpha(t) = 10$ rad/s$^2$. If the batter hits the ball 0.8 s later, what is the angular velocity? The (linear) speed of the part of the bat located 3 feet from the pivot point (the batter's body) is $v = 3\omega$. How fast is this part of the bat moving at the moment of contact? Through what angle was the bat rotated during the swing?

72. Suppose a golfer rotates a golf club through an angle of $3\pi/2$ with a constant angular acceleration of $\alpha$ rad/s$^2$. If the clubhead is located 4 feet from the pivot point (the golfer's body), the (linear) speed of the clubhead is $v = 4\omega$. Find the value of $\alpha$ that will produce a clubhead speed of 100 mph at impact.

73. From 1970 to 1974, the function $16.1e^{0.07t}$ closely approximated the number of millions of barrels of oil consumed per year in the United States, where $t = 0$ corresponds to 1970. When prices rose in 1974, the consumption rate was approximated by the function $21.3e^{0.04(t-4)}$. Show that both functions give (approximately) the same consumption rate for 1974. Compute $\int_4^{10} 16.1e^{0.07t}\,dt$ and $\int_4^{10} 21.3e^{0.04(t-4)}\,dt$ and compute the amount of oil saved by the reduced consumption.

74. The amount of work done by a force $F(x)$ moving an object from $x = a$ to $x = b$ is given by $\int_a^b F(x)\,dx$. As the object continues to move, the endpoint changes in time. Then $b = b(t)$ and the work done is a function of time. The derivative of this function defines **power.** Show that the power $\frac{d}{dt}\int_a^{b(t)} F(x)\,dx$ equals $F(b(t))\,b'(t)$.

**In exercises 75 and 76, use the result of exercise 74 to compute the horsepower in the following situations.**
**(Note: 1 hp = 550 ft-lb/s.)**

75. $F(x) = 1000$ lb, $b'(t) = 130$ ft/s

76. $F(x) = 1000e^{-x}$ lb, $b'(t) = 100 - t$ ft/s

**In exercises 77–82, find the average value of the function on the given interval.**

77. $f(x) = x^2 - 1,\ [1, 3]$

78. $f(x) = x^2 + 2x,\ [0, 1]$

79. $f(x) = 2x - 2x^2$, $[0, 1]$

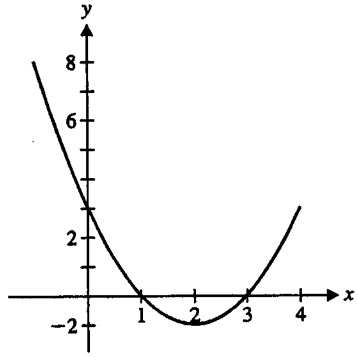
80. $f(x) = x^3 - 3x^2 + 2x$, $[1, 2]$

81. $f(x) = \cos x$, $[0, \pi/2]$

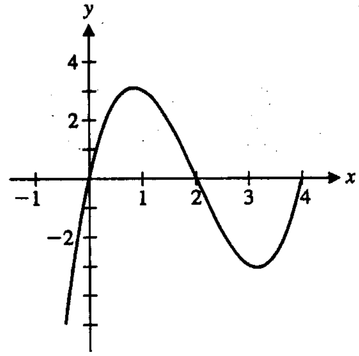
82. $f(x) = \sin x$, $[0, \pi/2]$

**In exercises 83 and 84, use the graph to list $\int_0^1 f(x)dx$, $\int_0^2 f(x)dx$ and $\int_0^3 f(x)dx$ in order, from smallest to largest.**

83.




84.




85. Derive **Leibniz' Rule:**

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))b'(x) - f(a(x))a'(x).$$

86. Suppose that a communicable disease has an infection stage and an incubation stage (like HIV and AIDS). Assume that the infection rate is a constant $f(t) = 100$ people per month and the incubation distribution is $b(t) = \frac{1}{900}te^{-t/30}$ month$^{-1}$. The rate at which people develop the disease at time $t = T$ is given by $r(T) = \int_0^T f(t)b(T - t)\,dt$ people per month. Use your CAS to find expressions for both the rate $r(T)$ and the number of people $p(x) = \int_0^x r(T)\,dT$ who develop the disease between times $t = 0$ and $t = x$. Explain why the graph $y = r(T)$ has a horizontal asymptote. For small $x$'s, the graph of $y = p(x)$ is concave up; explain what happens for large $x$'s. Repeat this for $f(t) = 100 + 10\sin t$, where the infection rate oscillates up and down.

87. When solving differential equations of the form $\frac{dy}{dt} = f(y)$ for the unknown function $y(t)$, it is often convenient to make use of a **potential function** $V(y)$. This is a function such that $-\frac{dV}{dy} = f(y)$. For the function $f(y) = y - y^3$, find a potential function $V(y)$. Find the locations of the local minima of $V(y)$ and use a graph of $V(y)$ to explain why this is called a "double-well" potential. Explain each step in the calculation

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = -f(y)\,f(y) \le 0.$$

Since $\frac{dV}{dt} \le 0$, does the function $V$ increase or decrease as time goes on? Use your graph of $V$ to predict the possible values of $\lim_{t\to\infty} y(t)$. Thus, you can predict the limiting value of the solution of the differential equation without ever solving the equation itself. Use this technique to predict $\lim_{t\to\infty} y(t)$ if $y' = 2 - 2y$.

## 4.6 INTEGRATION BY SUBSTITUTION

The Fundamental Theorem of Calculus gives us a powerful tool for evaluating definite integrals. The only problem is that to use it, we first need to find an antiderivative. Our goal in this section is to expand our ability to compute antiderivatives, through a useful technique called **integration by substitution.** As you have already seen, you need the chain rule to compute the derivatives of many functions. Integration by substitution gives us a process for helping to recognize when an integrand is the result of a chain rule derivative.