CAUTION

You must change the limits [illegible] integration as soon as you [illegible] variables!

We now have

$$\int_1^2 x^3\sqrt{x^4+5}\,dx = \frac{1}{4}\int_1^2 \underbrace{\sqrt{x^4+5}}_{\sqrt{u}}\,\underbrace{(4x^3)\,dx}_{du} = \frac{1}{4}\int_6^{21}\sqrt{u}\,du$$

$$= \frac{1}{4}\frac{u^{3/2}}{\left(\frac{3}{2}\right)}\Bigg|_6^{21} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)(21^{3/2}-6^{3/2}).$$

Notice that because we changed the limits of integration to match the new variable, we did **not** need to convert back to the original variable at the conclusion of the problem, as we do when we make a substitution in an indefinite integral. (Indeed, if we had switched the variables back, we would also have needed to switch the limits of integration back to their original values before evaluating!)

There is another way of dealing with definite integrals that may have occurred to you. You could always make a substitution to find an antiderivative and then return to the original variable to do the evaluation. Although this method will work for most of the problems you will encounter in this text, we recommend that you avoid it, for several reasons. First, changing the limits of integration is not very difficult and results in a much more readable mathematical expression. Second, in many applications of requiring substitution, you will **need** to change the limits of integration, so you might as well get used to doing so now.

Example 6.10 Substitution in a Definite Integral Involving an Exponential

Compute $\int_0^{15} te^{-t^2/2}\,dt$.

Solution As always, we look for terms that are derivatives of other terms. Here, you should notice that $\frac{d}{dt}\left(\frac{-t^2}{2}\right) = -t$. So, we set $u = -\frac{t^2}{2}$ and compute $du = -t\,dt$. For the upper limit of integration, we have that $t = 15$ corresponds to $u = -\frac{(15)^2}{2} = -\frac{225}{2}$. For the lower limit, we have that $t = 0$ corresponds to $u = 0$. This gives us

$$\int_0^{15} te^{-t^2/2}\,dt = -\int_0^{15}\underbrace{e^{-t^2/2}}_{e^u}\,\underbrace{(-t)\,dt}_{du}$$

$$= -\int_0^{-225/2} e^u\,du = -e^u\Big|_0^{-112.5} = -e^{-112.5}+1.$$

EXERCISES

1. It is never *wrong* to make a substitution in an integral, but sometimes it is not very helpful. For example, using the substitution $u = x^2$, you can correctly conclude that

$$\int x^3\sqrt{x^2+1}\,dx = \int \frac{1}{2}u\sqrt{u+1}\,du,$$

but the new integral is no easier than the original integral. In this case, a better substitution makes this workable. (Can you find it?) However, the general problem remains of how you can tell whether or not to give up on a substitution. Give some guidelines for answering this question, using the integrals $\int x\sin x^2\,dx$ and $\int x\sin x^3\,dx$ as illustrative examples.

2. It is not uncommon for students learning substitution to use incorrect notation in the intermediate steps. Be aware of this—it can be harmful to your grade! Carefully examine the following string of equalities and find each mistake. Using $u = x^2$,

$$\int_0^2 x \sin x^2\, dx = \int_0^2 (\sin u) x\, dx = \int_0^2 (\sin u) \frac{1}{2}\, du$$
$$= -\frac{1}{2} \cos u \Big|_0^2 = -\frac{1}{2} \cos x^2 \Big|_0^2$$
$$= -\frac{1}{2} \cos 4 + 1.$$

The final answer is correct, but because of several errors, this work would not earn full credit. Discuss each error and write this in a way that would earn full credit.

3. Suppose that an integrand has a term of the form $e^{f(x)}$. For example, suppose you are trying to evaluate $\int x^2 e^{x^3}\, dx$. Discuss why you should immediately try the substitution $u = f(x)$. If this substitution does not work, what could you try next? (Hint: Think about $\int x^2 e^{\ln x}\, dx$.)

4. Suppose that an integrand has a composite function of the form $f(g(x))$. Explain why you should look to see if the integrand also has the term $g'(x)$. Discuss possible substitutions.

In exercises 5–8, use the given substitution to evaluate the indicated integral.

5. $\int x^2 \sqrt{x^3 + 2}\, dx,\ u = x^3 + 2$

6. $\int x^3 (x^4 + 1)^{-2/3}\, dx,\ u = x^4 + 1$

7. $\int \frac{(\sqrt{x} + 2)^3}{\sqrt{x}}\, dx,\ u = \sqrt{x} + 2$

8. $\int \sin x \cos x\, dx,\ u = \sin x$

In exercises 9–46, evaluate the indicated integral.

9. $\int x(x^2 - 3)^4\, dx$

10. $\int x^3 \sqrt{x^4 + 3}\, dx$

11. $\int (2x + 1)(x^2 + x)^3\, dx$

12. $\int (x^2 + 2x)(x^3 + 3x^2)^2\, dx$

13. $\int \cos x \sqrt{\sin x + 1}\, dx$

14. $\int \sec^2 x \sqrt{\tan x}\, dx$

15. $\int \frac{\sin x}{\sqrt{\cos x}}\, dx$

16. $\int \sin^3 x \cos x\, dx$

17. $\int x^2 \cos x^3\, dx$

18. $\int \sec^2 x \cos(\tan x)\, dx$

19. $\int \sin x (\cos x + 3)^{3/4}\, dx$

20. $\int x \csc x^2 \cot x^2\, dx$

21. $\int x e^{x^2+1}\, dx$

22. $\int e^x \sqrt{e^x + 4}\, dx$

23. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

24. $\int \frac{e^{2x}}{(e^{2x} + 1)^3}\, dx$

25. $\int \frac{x^2}{\sqrt{x^3 - 2}}\, dx$

26. $\int \frac{x + 1}{(x^2 + 2x - 1)^2}\, dx$

27. $\int \frac{(\ln x + 2)^2}{x}\, dx$

28. $\int \frac{\sqrt{\ln x}}{x}\, dx$

29. $\int \frac{2x + 1}{x^2 + x - 1}\, dx$

30. $\int \frac{\cos x}{\sin x + 2}\, dx$

31. $\int \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2}\, dx$

32. $\int \frac{x}{x^2 + 4}\, dx$

33. $\int \cos x\, e^{\sin x}\, dx$

34. $\int \tan 2x\, dx$

35. $\int \sin x (\cos x - 1)^3\, dx$

36. $\int x^2 \sec^2 x^3\, dx$

37. $\int \frac{4}{x(\ln x + 1)^2}\, dx$

38. $\int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx$

39. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

40. $\int \frac{6x}{(x^2 - 3)^4}\, dx$

41. $\int \frac{1}{x \ln \sqrt{x}}\, dx$

42. $\int \frac{\cos x}{\sin^2 x}\, dx$

43. $\int \frac{2x + 3}{x + 7}\, dx$

44. $\int \frac{3x + 4}{2x + 7}\, dx$

45. $\int \frac{x^2}{\sqrt[3]{x + 3}}\, dx$

46. $\int \frac{x^2 + 2}{\sqrt{x - 5}}\, dx$

In exercises 47–56, evaluate the definite integral.

47. $\int_0^2 x \sqrt{x^2 + 1}\, dx$

48. $\int_1^3 x \sin(\pi x^2)\, dx$

49. $\int_{-1}^1 \frac{x}{(x^2 + 1)^2}\, dx$

50. $\int_0^2 x^2 e^{x^3}\, dx$

51. $\int_{\pi/2}^{\pi} \frac{4\cos x}{(\sin x+1)^2}\,dx$

52. $\int_0^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

53. $\int_{\pi/4}^{\pi/2} \cot x\,dx$

54. $\int_1^e \frac{\ln x}{x}\,dx$

55. $\int_1^4 \frac{x-1}{\sqrt{x}}\,dx$

56. $\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$

In exercises 57–66, evaluate the integral exactly, if possible. Otherwise, estimate it numerically.

57. $\int_0^{\pi} \sin x^2\,dx$

58. $\int_0^{\pi} x\sin x^2\,dx$

59. $\int_{-1}^1 xe^{-x^2}\,dx$

60. $\int_{-1}^1 e^{-x^2}\,dx$

61. $\int_0^2 \frac{4}{(x^2+1)^2}\,dx$

62. $\int_0^2 \frac{4x}{(x^2+1)^2}\,dx$

63. $\int_0^2 \frac{4x^2}{(x^2+1)^2}\,dx$

64. $\int_0^2 \frac{4x^3}{(x^2+1)^2}\,dx$

65. $\int_0^{\pi/4} \sec x\,dx$

66. $\int_0^{\pi/4} \sec^2 x\,dx$

In exercises 67–70, make the indicated substitution for an unspecified function $f(x)$.

67. $u=x^2$ for $\int_0^2 xf(x^2)\,dx$

68. $u=x^3$ for $\int_1^2 x^2f(x^3)\,dx$

69. $u=\sin x$ for $\int_0^{\pi/2} (\cos x)f(\sin x)\,dx$

70. $u=\sqrt{x}$ for $\int_0^4 \frac{f(\sqrt{x})}{\sqrt{x}}\,dx$

71. A function $f(x)$ is said to be **even** if $f(-x)=f(x)$ for all x. $f(x)=x^2$ and $f(x)=x^4$ are even, since $(-x)^2=x^2$ and $(-x)^4=x^4$. A function $f(x)$ is said to be **odd** if $f(-x)=-f(x)$. Show that $\cos x$ and $x\sin x$ are even, but $\sin x$ and $x\cos x$ are odd.

72. Suppose $f(x)$ is continuous for all x. For any positive constant a, $\int_{-a}^a f(x)\,dx=\int_{-a}^0 f(x)\,dx+\int_0^a f(x)\,dx$. Using the substitution $u=-x$ in $\int_{-a}^0 f(x)\,dx$, show that if $f(x)$ is even, then $\int_{-a}^a f(x)\,dx=2\int_0^a f(x)\,dx$. Also, if $f(x)$ is odd, show that $\int_{-a}^a f(x)\,dx=0$.

In exercises 73–78, determine if the integrand is even or odd (or neither), rewrite the integral accordingly and compute (or estimate) the integral.

73. $\int_{-1}^1 x\cos x\,dx$

74. $\int_{-1}^1 x^2\sin x\,dx$

75. $\int_{-1}^1 (x^4-2x^2+1)\,dx$

76. $\int_{-1}^1 xe^{-x^2}\,dx$

77. $\int_{-1}^1 (x^2+\sin x)\,dx$

78. $\int_{-1}^1 (x^3-2x)\,dx$

79. The location $(\bar{x},\bar{y})$ of the center of gravity (balance point) of a flat plate bounded by $y=f(x)>0$, $a\le x\le b$ and the x-axis is given by $\bar{x}=\frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}$ and $\bar{y}=\frac{\int_a^b [f(x)]^2\,dx}{2\int_a^b f(x)\,dx}$. For the semicircle $y=f(x)=\sqrt{4-x^2}$, use symmetry as in exercises 73–78 to argue that $\bar{x}=0$ and $\bar{y}=\frac{1}{2\pi}\int_0^2(4-x^2)\,dx$. Compute $\bar{y}$.

80. Suppose that the population density of a group of animals can be described by $f(x)=xe^{-x^2}$ thousand animals per mile for $0\le x\le 2$, where x is the distance from a pond. Graph $y=f(x)$ and briefly describe where these animals are likely to be found. Find the total population $\int_0^2 f(x)\,dx$.

81. The voltage in an AC (alternating current) circuit is given by $V(t)=V_p\sin(2\pi ft)$, where f is the frequency. A voltmeter does not indicate the amplitude V_p. Instead, the voltmeter reads the **root-mean-square** (rms), the square root of the average value of the square of the voltage over one cycle. That is, rms $=\sqrt{f\int_0^{1/f}V^2(t)\,dt}$. Use the trigonometric identity $\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$ to show that rms $=V_p/\sqrt{2}$.

82. Graph $y=f(t)$ and find the root-mean-square of

$$f(t)=\begin{cases}-1 & \text{if } -2\le t<-1\\ t & \text{if } -1\le t\le 1\\ 1 & \text{if } 1<t\le 2\end{cases},\ \text{where rms}=\sqrt{\frac{1}{4}\int_{-2}^2 f^2(t)\,dt}.$$

83. A **predator-prey system** is a set of differential equations modeling the change in population of interacting species of organisms. A simple model of this type is

$$\begin{cases}x'(t)=x(t)[a-by(t)]\\ y'(t)=y(t)[dx(t)-c]\end{cases}$$

for positive constants a, b, c and d. Each equation includes a term of the form $x(t)y(t)$, which is intended to represent the result of confrontations between the species. Noting that the contribution of this term is negative to $x'(t)$ but positive to $y'(t)$, explain why it must be that $x(t)$ represents the population of the prey and $y(t)$ the population of the predator. If $x(t)=y(t)=0$, compute $x'(t)$ and $y'(t)$. In this case, will

x and y increase, decrease or stay constant? Explain why this makes sense physically. Determine $x'(t)$ and $y'(t)$ and the subsequent change in x and y at the so-called **equilibrium point** $x = c/d$, $y = a/b$. If the population is periodic, we can show that the equilibrium point gives the average population (even if the population does not remain constant). To do so, note that $\frac{x'(t)}{x(t)} = a - by(t)$. Integrating both sides of this equation from $t = 0$ to $t = T$ [the period of $x(t)$ and $y(t)$], we get $\int_0^T \frac{x'(t)}{x(t)}\,dt = \int_0^T a\,dt - \int_0^T by(t)\,dt$. Evaluate each integral to show that $\ln x(T) - \ln x(0) = aT - \int_0^T by(t)\,dt$. Assuming that $x(t)$ has period T, we have $x(T) = x(0)$ and so, $0 = aT - \int_0^T by(t)\,dt$. Finally, rearrange terms to show that $1/T \int_0^T y(t)\,dt = a/b$; that is, the average value of the population $y(t)$ is the equilibrium value $y = a/b$. Similarly, show that the average value of the population $x(t)$ is the equilibrium value $x = c/d$.

4.7 NUMERICAL INTEGRATION

Thus far, our development of the integral has paralleled our development of the derivative. In both cases, we began with a limit definition that was difficult to use for calculation and then, proceeded to develop simplified rules for calculation. At this point, you should be able to find the derivative of nearly any function you can write down. You might expect that with a few more rules you will be able to do the same for integrals. Unfortunately, this is not the case. There are many functions for which **no** elementary antiderivative is available. (By elementary antiderivative, we mean an antiderivative expressible in terms of the elementary functions with which you are familiar: the algebraic, trigonometric, exponential and logarithmic functions.) For instance,

$$\int_0^2 \cos(x^2)\,dx$$

cannot be calculated exactly, since $\cos(x^2)$ does not have an elementary antiderivative. (Try to find one, but don't spend much time, as mathematicians have proven that it can't be done.)

In fact, most definite integrals cannot be calculated exactly. When we can't compute the value of an integral exactly, we do the next best thing: we approximate its value numerically. In this section, we develop three methods of approximating definite integrals. None will replace the built-in integration routine on your calculator or computer. However, by exploring these methods, you will be exposed to some of the basic ideas used to develop more sophisticated numerical integration routines.

You should recognize that you already have a number of approximation methods at your disposal. Since a definite integral is the limit of a sequence of Riemann sums, any Riemann sum serves as an approximation of the integral,

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i is any point chosen from the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. From the definition of definite integral, observe that the larger n is, the better the approximation tends to be. The reason we say that Riemann sums provide us with numerous approximation schemes is that we are free to choose the evaluation points, c_i, for $i = 1, 2, \ldots, n$. The most common choice leads to a method called the **Midpoint Rule:**

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x,$$