If we require the above bound on the error to be no larger than the required accuracy of $10^{-7}$, we have

$$|ET_n| \leq \frac{4}{3n^2} \leq 10^{-7}.$$

Solving this inequality for $n^2$ gives us

$$\frac{4}{3}10^7 \leq n^2$$

and taking the square root of both sides yields

$$n \geq \sqrt{\frac{4}{3}10^7} \approx 3651.48.$$

So, any value of $n \geq 3652$ will give the required accuracy. Similarly, for Simpson's Rule, we have

$$|ES_n| \leq L\frac{(b-a)^5}{180n^4} = 24\frac{(3-1)^5}{180n^4}.$$

Again, requiring that the error bound be no larger than $10^{-7}$, gives us

$$|ES_n| \leq 24\frac{(3-1)^5}{180n^4} \leq 10^{-7}$$

and solving for $n^4$, we have

$$n^4 \geq 24\frac{(3-1)^5}{180}10^7.$$

Upon taking fourth roots, we get

$$n \geq \sqrt[4]{24\frac{(3-1)^5}{180}10^7} \approx 80.8,$$

so that taking any value of $n \geq 82$ will guarantee the required accuracy. (If you expected us to say that $n \geq 81$, keep in mind that Simpson's Rule requires $n$ to be even.) ■

In example 7.11, compare the number of steps required to guarantee $10^{-7}$ accuracy in Simpson's Rule (82) to the number required to guarantee the same accuracy in Trapezoidal Rule (3652). This is typically the case, that Simpson's Rule requires far fewer steps than either Trapezoidal Rule or Midpoint Rule to get the same accuracy. Finally, from example 7.11, observe that we now know that

$$\ln 3 = \int_1^3 \frac{1}{x}\,dx \approx S_{82} \approx 1.0986123,$$

which is guaranteed (by Theorem 7.2) to be correct to within $10^{-7}$. Compare this with the approximate value of ln 3 generated by your calculator.

## EXERCISES 4.7

1. Ideally, approximation techniques should be both simple and accurate. How do the numerical integration methods presented in this section compare in terms of simplicity and accuracy? Which criterion would be more important if you were working entirely by hand? Which method would you use? Which criterion would be more important if you were using a very fast computer? Which method would you use?

2. Suppose you were going to construct your own rule for approximate integration. (Name it after yourself!) In the

text, new methods were obtained both by choosing evaluation points for Riemann sums (Midpoint Rule) and by geometric construction (Trapezoidal Rule and Simpson's Rule). Without working out the details, explain how you would develop a very accurate but simple rule.

3. Test your calculator or computer on $\int_0^1 \sin(1/x)\,dx$. Discuss what your options are when your technology does not immediately return an accurate approximation. Based on a quick sketch of $y = \sin(1/x)$, describe why a numerical integration routine would have difficulty with this integral.

4. Explain why we did not use the Midpoint Rule in example 7.4.

**In exercises 5–8, compute Midpoint, Trapezoidal and Simpson's Rule approximations by hand (leave your answer as a fraction) for $n = 4$.**

5. $\int_0^1 (x^2 + 1)\,dx$

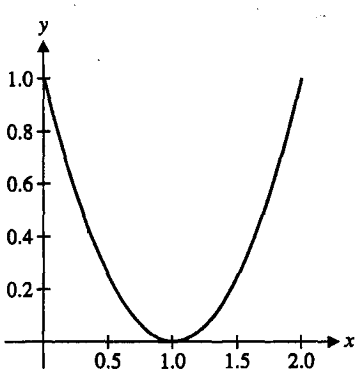
6. $\int_0^2 (x^2 + 1)\,dx$

7. $\int_1^3 \frac{1}{x}\,dx$

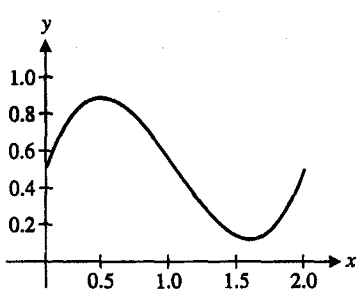
8. $\int_{-1}^1 (2x - x^2)\,dx$

**In exercises 9 and 10, use the graph to estimate (a) Riemann sum with left-endpoint evaluation, (b) Midpoint Rule and (c) Trapezoidal Rule approximations with $n = 4$ of $\int_0^2 f(x)dx$.**

9.




10.




**In exercises 11–16, use a computer or calculator to compute the Midpoint, Trapezoidal and Simpson's Rule approximations with $n = 10$, $n = 20$ and $n = 50$. Compare these values to the approximation given by your calculator or computer.**

11. $\int_0^\pi \cos x^2\,dx$

12. $\int_0^{\pi/4} \sin \pi x^2\,dx$

13. $\int_0^2 e^{-x^2}\,dx$

14. $\int_0^3 e^{-x^2}\,dx$

15. $\int_0^\pi e^{\cos x}\,dx$

16. $\int_0^1 \sqrt[3]{x^2 + 1}\,dx$

**In exercises 17–20, compute the exact value and compute the error (the difference between the approximation and the exact value) in each of the Midpoint, Trapezoidal and Simpson's Rule approximations using $n = 10$, $n = 20$, $n = 40$ and $n = 80$.**

17. $\int_0^1 5x^4\,dx$

18. $\int_1^2 \frac{1}{x}\,dx$

19. $\int_0^\pi \cos x\,dx$

20. $\int_0^{\pi/4} \cos x\,dx$

21. Fill in the blanks with the most appropriate power of 2 (2, 4, 8, etc.). If you double $n$, the error in the Midpoint Rule is divided by______. If you double $n$, the error in the Trapezoidal Rule is divided by______. If you double $n$, the error in Simpson's Rule is divided by______.

22. Fill in the blanks with the most appropriate power of 2 (2, 4, 8, etc.). If you halve the interval length $b - a$, the error in the Midpoint Rule is divided by______, the error in the Trapezoidal Rule is divided by______ and the error in Simpson's Rule is divided by______.

**In exercises 23–26, approximate the given value using (a) Midpoint Rule, (b) Trapezoidal Rule and (c) Simpson's Rule with $n = 4$.**

23. $\ln 4 = \int_1^4 \frac{1}{x}\,dx$

24. $\ln 8 = \int_1^8 \frac{1}{x}\,dx$

25. $\sin 1 = \int_0^1 \cos x\,dx$

26. $e^2 = \int_0^1 (2e^{2x} + 1)\,dx$

27. For exercise 23, find the number of steps needed to guarantee an accuracy of $10^{-7}$.

28. For exercise 25, find the number of steps needed to guarantee an accuracy of $10^{-7}$.

29. For each rule in exercise 17, compute the error bound and compare it to the actual error.

30. For each rule in exercise 19, compute the error bound and compare it to the actual error.

**In exercises 31–34, use (a) Trapezoidal Rule and (b) Simpson's Rule to estimate $\int_0^2 f(x)dx$ from the given data.**

31.

| $x$ | 0.0 | 0.25 | 0.5 | 0.75 | 1.0 |
|---|---|---|---|---|---|
| $f(x)$ | 4.0 | 4.6 | 5.2 | 4.8 | 5.0 |

| $x$ | 1.25 | 1.5 | 1.75 | 2.0 |
|---|---|---|---|---|
| $f(x)$ | 4.6 | 4.4 | 3.8 | 4.0 |

32.

| $x$ | 0.0 | 0.25 | 0.5 | 0.75 | 1.0 |
|---|---|---|---|---|---|
| $f(x)$ | 1.0 | 0.6 | 0.2 | −0.2 | −0.4 |

| $x$ | 1.25 | 1.5 | 1.75 | 2.0 |
|---|---|---|---|---|
| $f(x)$ | 0.4 | 0.8 | 1.2 | 2.0 |

33.

| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $f(x)$ | 2.4 | 2.6 | 2.9 | 3.2 | 3.4 | 3.6 |

| $x$ | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|
| $f(x)$ | 3.8 | 3.9 | 4.0 | 4.1 | 4.2 |

34.

| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $f(x)$ | 1.2 | 0.8 | 0.4 | 0.2 | −0.4 | −0.6 |

| $x$ | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|
| $f(x)$ | −0.4 | −0.2 | 0.0 | 0.4 | 1.2 |

**In exercises 35 and 36, the table gives the measurements (in feet) of the width of a plot of land at 10-foot intervals. Estimate the area of the plot.**

35.

| $x$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 56 | 54 | 58 | 62 | 58 | 58 | 62 |

| $x$ | 70 | 80 | 90 | 100 | 110 | 120 |
|---|---|---|---|---|---|---|
| $f(x)$ | 56 | 52 | 48 | 40 | 32 | 22 |

36.

| $x$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 26 | 30 | 28 | 22 | 28 | 32 | 30 |

| $x$ | 70 | 80 | 90 | 100 | 110 | 120 |
|---|---|---|---|---|---|---|
| $f(x)$ | 33 | 31 | 28 | 30 | 32 | 22 |

**In exercises 37 and 38, the velocity of an object at various times is given. Use the data to estimate the distance traveled.**

37.

| $t$ (s) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $v(t)$ (ft/s) | 40 | 42 | 40 | 44 | 48 | 50 | 46 |

| $t$ (s) | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|
| $v(t)$ (ft/s) | 46 | 42 | 44 | 40 | 42 | 42 |

38.

| $t$ (s) | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $v(t)$ (ft/s) | 26 | 30 | 28 | 30 | 28 | 32 | 30 |

| $t$ (s) | 14 | 16 | 18 | 20 | 22 | 24 |
|---|---|---|---|---|---|---|
| $v(t)$ (ft/s) | 33 | 31 | 28 | 30 | 32 | 32 |

**In exercises 39 and 40, the data comes from a pneumotachograph which measures air flow through the throat (in liters per second). The integral of the air flow equals the volume of air exhaled. Estimate this volume.**

39.

| $t$ (s) | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
|---|---|---|---|---|---|---|---|
| $f(t)$ (l/s) | 0 | 0.2 | 0.4 | 1.0 | 1.6 | 2.0 | 2.2 |

| $t$ (s) | 1.4 | 1.6 | 1.8 | 2.0 | 2.2 | 2.4 |
|---|---|---|---|---|---|---|
| $f(t)$ (l/s) | 2.0 | 1.6 | 1.2 | 0.6 | 0.2 | 0 |

40.

| $t$ (s) | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
|---|---|---|---|---|---|---|---|
| $f(t)$ (l/s) | 0 | 0.1 | 0.4 | 0.8 | 1.4 | 1.8 | 2.0 |

| $t$ (s) | 1.4 | 1.6 | 1.8 | 2.0 | 2.2 | 2.4 |
|---|---|---|---|---|---|---|
| $f(t)$ (l/s) | 2.0 | 1.6 | 1.0 | 0.6 | 0.2 | 0 |

**In exercises 41–46, use the given information about $f(x)$ and its derivatives to determine whether (a) the Midpoint Rule would be exact, underestimate or overestimate the integral (or there's not enough information to tell). Repeat for (b) Trapezoidal Rule and (c) Simpson's Rule.**

41. $f''(x) > 0, f'(x) > 0$

42. $f''(x) > 0, f'(x) < 0$

43. $f''(x) < 0, f'(x) > 0$

44. $f''(x) < 0, f'(x) < 0$

45. $f''(x) = 4, f'(x) > 0$

46. $f''(x) = 0, f'(x) > 0$

47. Sketch a trapezoid with base $L$ and heights $h_1$ and $h_2$. Show that the area of the trapezoid is given by $L(h_1 + h_2)/2$.

48. Suppose that $R_L$ and $R_R$ are the Riemann sum approximations of $\int_a^b f(x)\,dx$ using left- and right-endpoint evaluation rules, respectively, for some $n > 0$. Show that the trapezoidal approximation $T_n$ is equal to $(R_L + R_R)/2$.

49. For the data in Figure 4.30, sketch in the two approximating parabolas for Simpson's Rule. Compare the Simpson's Rule approximation to the Trapezoidal Rule approximation. Explain graphically why the Simpson's Rule approximation is smaller.

50. For the data in Figure 4.30, sketch a smooth curve that passes through the five data points. Which more closely approximates your smooth curve, the line segments in Figure 4.31 or the parabolas you drew in exercise 49? Would your answer change if you had twice as many data points?

51. In most of the calculations that you have done, it is true that the Trapezoidal Rule and Midpoint Rule are on opposite sides of the exact integral (i.e., one is too large, the other too small). Also, you may have noticed that the Trapezoidal Rule tends to be about twice as far from the exact value as the Midpoint Rule. Given this, explain why the linear combination $\frac{1}{3}T_n + \frac{2}{3}M_n$ should give a good estimate of the integral. (Here, $T_n$ represents the Trapezoidal Rule approximation using $n$ partitions and $M_n$ the corresponding Midpoint Rule approximation.)

52. Show that the approximation rule $\frac{1}{3}T_n + \frac{2}{3}M_n$ in exercise 51 is identical to Simpson's Rule.

53. Compute the Trapezoidal Rule approximations $T_4$, $T_8$ and $T_{16}$ of $\int_0^1 3x^2\,dx$ and compute the error (the difference between the approximation and the exact value of 1). Verify that when the step size is cut in half, the error is divided by four. When such patterns emerge, they can be taken advantage of using **extrapolation.** The idea is simple: if the approximations continually get smaller, then the value of the integral is smaller and we should be able to predict (extrapolate) how much smaller the integral is. Given that $(T_4 - I) = 4(T_8 - I)$, where $I = 1$ is the exact integral, show that $I = T_8 + \dfrac{T_8 - T_4}{3}$. Also, show that $I = T_{16} + \dfrac{T_{16} - T_8}{3}$. In general, we have the *approximations* $(T_4 - I) \approx 4(T_8 - I)$ and $I \approx T_8 + \dfrac{T_8 - T_4}{3}$. Then the extrapolation $E_{2n} = T_{2n} + \dfrac{T_{2n} - T_n}{3}$ is closer to the exact integral than either of the individual Trapezoidal Rule approximations $T_{2n}$ and $T_n$. Show that, in fact, $E_{2n}$ equals the Simpson's Rule approximation for $2n$.

54. The geometric construction of Simpson's Rule makes it clear that Simpson's Rule will compute integrals such as $\int_0^1 3x^2\,dx$ exactly. Briefly explain why. Now, compute Simpson's Rule with $n = 2$ for $\int_0^1 4x^3\,dx$. It turns out that Simpson's Rule also computes integrals of cubics exactly! In this exercise, we want to understand why a method that uses parabolas can compute integrals of cubics exactly. But first, sketch out the Midpoint Rule approximation of $\int_0^1 2x\,dx$ with $n = 1$. On part of the interval, the midpoint rectangle is above the straight line and on part of the interval, the midpoint rectangle is below the line. Explain why the Midpoint Rule computes the area exactly. Now, back to Simpson's Rule. To see how Simpson's Rule works on $\int_0^1 4x^3\,dx$, we need to determine the actual parabola being used. The parabola must pass through the points $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(1, 4)$. Find the quadratic function $y = ax^2 + bx + c$ that accomplishes this. (Hint: Explain why $0 = 0 + 0 + c$, $\dfrac{1}{2} = \dfrac{a}{4} + \dfrac{b}{2} + c$ and $4 = a + b + c$ and then solve for $a$, $b$ and $c$.) Graph this parabola and $y = 4x^3$ on the same axes, carefully choosing the graphing window so that you can see what is happening on the interval $[0, 1]$. Where is the vertex of the parabola? How do the integrals of the parabola and cubic compare on the subinterval $\left[0, \frac{1}{2}\right]$? $\left[\frac{1}{2}, 1\right]$? Why does Simpson's Rule compute the integral exactly?

# CHAPTER REVIEW EXERCISES

5. $\int 2\sin 4x\,dx$

6. $\int 3\sec^2 x\,dx$

13. $\int x\sqrt{x^2+4}\,dx$

14. $\int x(x^2+4)\,dx$