Example 1.7 **Estimating the Energy Lost by a Tennis Ball**

Suppose that test measurements provide the following data on the collision of a tennis ball with a racket. Estimate the percentage of energy lost in the collision.

x (in.)	0.0	0.1	0.2	0.3	0.4
$f_c(x)$ (lb)	0	25	50	90	160
$f_e(x)$ (lb)	0	23	46	78	160

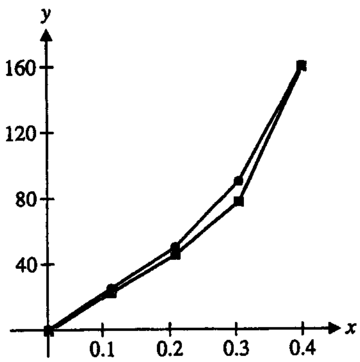


Figure 5.11
Force exerted on a tennis ball.

Solution The data are plotted in Figure 5.11, shown with line segments connecting the data points.

We need to estimate the area between the curves and the area under the top curve. Since we don't have a formula for either function, we must use a numerical method such as Simpson's Rule. For $\int_0^{0.4} f_c(x)\,dx$, we get

$$\int_0^{0.4} f_c(x)\,dx \approx \frac{0.1}{3}[0 + 4(25) + 2(50) + 4(90) + 160] = 24.$$

To use Simpson's Rule to approximate $\int_0^{0.4} [f_c(x) - f_e(x)]\,dx$, we need a table of function values for $f_c(x) - f_e(x)$. Subtraction gives us

x	0.0	0.1	0.2	0.3	0.4
$f_c(x) - f_e(x)$	0	2	4	12	0

from which Simpson's Rule gives us

$$\int_0^{0.4} [f_c(x) - f_e(x)]\,dx \approx \frac{0.1}{3}[0 + 4(2) + 2(4) + 4(12) + 0] = \frac{6.4}{3}.$$

The percentage of energy lost is then $\frac{100(6.4/3)}{24} = 8.9\%$. With over 90% of its energy retained in the collision, this is a lively tennis ball.

EXERCISES

1. Suppose the functions f and g satisfy $f(x) \geq g(x) \geq 0$ for all x in the interval $[a, b]$. Explain in terms of the areas $A_1 = \int_a^b f(x)\,dx$ and $A_2 = \int_a^b g(x)\,dx$ why the area between the curves $y = f(x)$ and $y = g(x)$ is given by $\int_a^b |f(x) - g(x)|\,dx$.

2. Suppose the functions f and g satisfy $f(x) \leq g(x) \leq 0$ for all x in the interval $[a, b]$. Explain in terms of the areas $A_1 = \int_a^b f(x)\,dx$ and $A_2 = \int_a^b g(x)\,dx$ why the area between the curves $y = f(x)$ and $y = g(x)$ is given by $\int_a^b |f(x) - g(x)|\,dx$.

3. Suppose that the speeds of racing cars A and B are $v_A(t)$ and $v_B(t)$ mph, respectively. If $v_A(t) \geq v_B(t)$ for all t, $v_A(0) = v_B(0)$ and the race lasts from $t = 0$ to $t = 2$ hours, explain why car A will win the race by $\int_0^2 [v_A(t) - v_B(t)]\,dt$ miles.

4. Suppose that the speeds of racing cars A and B are $v_A(t)$ and $v_B(t)$ mph, respectively. If $v_A(t) \geq v_B(t)$ for $0 \leq t \leq 0.5$ and $1.1 \leq t \leq 1.6$ and $v_B(t) \geq v_A(t)$ for $0.5 \leq t \leq 1.1$ and $1.6 \leq t \leq 2$, describe the difference between $\int_0^2 |V_A(t) - V_B(t)|\,dt$ and $\int_0^2 [V_A(t) - V_B(t)]\,dt$. Which integral will tell you which car wins the race?

In exercises 5–12, find the area between the curves on the given interval.

5. $y = x^3, y = x^2 - 1, 1 \leq x \leq 3$

6. $y = \cos x, y = x^2 + 2, 0 \leq x \leq 2$

7. $y = e^x, y = x - 1, -2 \leq x \leq 0$

8. $y = e^{-x}, y = x^2, 1 \leq x \leq 4$

9. $y = x^2 - 1, y = 1 - x, 0 \leq x \leq 2$

10. $y = x^2 - 3, y = x - 1, 0 \leq x \leq 3$

11. $y = x^3 - 1, y = 1 - x, -2 \leq x \leq 2$

12. $y = x^4 + x - 2, y = x - 1, -2 \leq x \leq 2$

In exercises 13–20, sketch and find the area of the region determined by the intersections of the curves.

13. $y = x^2 - 1, y = 7 - x^2$

14. $y = x^2 - 1, y = \frac{1}{2}x^2$

15. $y = x^2 + 1, y = 3x - 1$

16. $y = x^2 - x - 4, y = x + 4$

17. $y = x^3, y = 3x + 2$

18. $y = x^3 - 2x^2, y = x^2$

19. $y = x^3, y = x^2$

20. $y = \sqrt{x}, y = x^2$

In exercises 21–26, sketch and estimate the area determined by the intersections of the curves.

21. $y = e^x, y = 1 - x^2$

22. $y = x^4, y = 1 - x$

23. $y = \sin x, y = x^2$

24. $y = \cos x, y = x^4$

25. $y = x^4, y = 2 + x$

26. $y = \ln x, y = x^2 - 2$

In exercises 27–34, sketch and find the area of the region bounded by the given curves. Choose the variable of integration so that the area is written as a single integral.

27. $y = x, y = 2 - x, y = 0$

28. $y = 2x\ (x > 0), y = 3 - x^2, x = 0$

29. $x = 3y, x = 2 + y^2$

30. $x = y^2, x = 1$

31. $x = y, x = -y, x = 1$

32. $y = x, y = -x, y = 2$

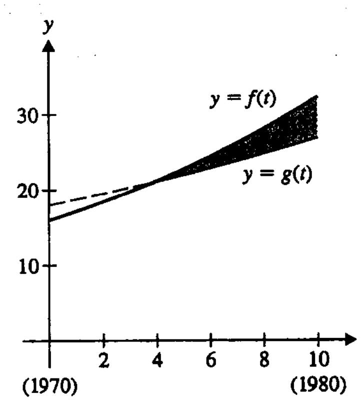
33. $y = x, y = 2, y = 6 - x, y = 0$

34. $x = y^2, x = 4$

35. The average value of a function $f(x)$ on the interval $[a, b]$ is $A = \frac{1}{b-a}\int_a^b f(x)\,dx$. Compute the average value of $f(x) = x^2$ on $[0, 3]$ and show that the area above $y = A$ and below $y = f(x)$ equals the area below $y = A$ and above the x-axis.

36. Prove that the result of exercise 35 is always true by showing that $\int_a^b [f(x) - A]\,dx = 0$.

37. The United States oil consumption for the years 1970–1974 was approximately equal to $f(t) = 16.1e^{0.07t}$ million barrels per year, where $t = 0$ corresponds to 1970. Following an oil shortage in 1974, the country's consumption changed and was better modeled by $g(t) = 21.3e^{0.04(t-4)}$ million barrels per year, $t \geq 4$. Show that $f(4) \approx g(4)$ and explain what this number represents. Compute the area between $f(t)$ and $g(t)$ for $4 \leq t \leq 10$. Use this number to estimate the number of barrels of oil saved by Americans' reduced oil consumption from 1974 to 1980.



38. Suppose that a nation's fuelwood consumption is given by $76e^{0.03t}$ m^3/yr and new tree growth is $50 - 6e^{0.09t}$ m^3/yr. Compute and interpret the area between the curves for $0 \leq t \leq 10$.

39. Suppose that the birth rate for a certain population is $b(t) = 2e^{0.04t}$ million people per year, and the death rate for the same population is $d(t) = 2e^{0.02t}$ million people per year. Show that $b(t) \geq d(t)$ for $t \geq 0$, and explain why the area between the curves represents the increase in population. Compute the increase in population for $0 \leq t \leq 10$.

40. Suppose that the birth rate for a population is $b(t) = 2e^{0.04t}$ million people per year, and the death rate for the same population is $d(t) = 3e^{0.02t}$ million people per year. Find the intersection T of the curves ($T > 0$). Interpret the area between the curves for $0 \le t \le T$ and the area between the curves for $T \le t \le 30$. Compute the net change in population for $0 \le t \le 30$.

41. In collisions between a ball and a striking object (e.g., a baseball bat or tennis racket), the ball changes shape, first compressing and then expanding. If x represents the change in size of the ball (e.g., in inches) for $0 \le x \le m$ and $f(x)$ represents the force between the ball and striking object (e.g., in pounds), the area under the curve $y = f(x)$ is proportional to the energy transferred. Suppose that $f_c(x)$ is the force during compression and $f_e(x)$ is the force during expansion. Explain why $\int_0^m [f_c(x) - f_e(x)]\,dx$ is proportional to the energy lost by the ball (due to friction) and thus $\int_0^m [f_c(x) - f_e(x)]\,dx / \int_0^m f_c(x)\,dx$ is the proportion of energy lost in the collision. For a baseball and bat, reasonable values are shown (see Adair's book *The Physics of Baseball*):

x (in.)	0	0.1	0.2	0.3	0.4
$f_c(x)$ (lb)	0	250	600	1200	1750
$f_e(x)$ (lb)	0	10	100	270	1750

Use Simpson's Rule to estimate the proportion of energy retained by the baseball.

42. Using the same notation as in exercise 41, values for the force $f_c(x)$ during compression and force $f_e(x)$ during expansion of a golf ball are given by

x (in.)	0	0.045	0.09	0.135	0.18
$f_c(x)$ (lb)	0	200	500	1000	1800
$f_e(x)$ (lb)	0	125	350	700	1800

Use Simpson's Rule to estimate the proportion of energy retained by the golf ball.

43. Much like the compression and expansion of a ball discussed in exercises 41 and 42, the force exerted by a tendon as a function of its extension determines the loss of energy. Suppose that x is the extension of the tendon, $f_s(x)$ is the force during stretching of the tendon and $f_r(x)$ is the force during recoil of the tendon. The data given is for a hind leg tendon of a wallaby (see Alexander's book *Exploring Biomechanics*):

x (mm)	0	0.75	1.5	2.25	3.0
$f_s(x)$ (N)	0	110	250	450	700
$f_r(x)$ (N)	0	100	230	410	700

Use Simpson's Rule to estimate the proportion of energy returned by the tendon.

44. The arch of a human foot acts like a spring during walking and jumping, storing energy as the foot stretches (i.e., the arch flattens) and returning energy as the foot recoils. In the data, x is the vertical displacement of the arch, $f_s(x)$ is the force on the foot during stretching and $f_r(x)$ is the force during recoil (see Alexander's book *Exploring Biomechanics*):

x (mm)	0	2.0	4.0	6.0	8.0
$f_s(x)$ (N)	0	300	1000	1800	3500
$f_r(x)$ (N)	0	150	700	1300	3500

Use Simpson's Rule to estimate the proportion of energy returned by the arch.

45. The velocities $f(t)$ and $g(t)$ of two falling objects are given by $f(t) = -40 - 32t$ ft/s and $g(t) = -30 - 32t$ ft/s. Assume that the objects start at the same height at time $t = 0$. Find and interpret the area between the curves for $0 \le t \le 10$.

46. The velocities of two runners are given by $f(t) = 10$ mph and $g(t) = 10 - \sin t$ mph. Find and interpret the integrals $\int_0^{\pi} [f(t) - g(t)]\,dt$ and $\int_0^{2\pi} [f(t) - g(t)]\,dt$.

47. The velocities of two racing cars A and B are given by $f(t) = 40(1 - e^{-t})$ mph and $g(t) = 20t$ mph, respectively. The cars start at the same place at time $t = 0$. Estimate (a) the largest lead for car A and (b) the time at which car B catches up.

48. At this stage, you can compute the area of any "simple" planar region. For a general figure bounded on the left by a function $x = l(y)$, on the right by a function $x = r(y)$, on top by a function $y = t(x)$ and on the bottom by a function $y = b(x)$, write the area as a sum of integrals. (Hint: Divide the region into subregions whose area can be written as the integral of $r(y) - l(y)$ or $t(x) - b(x)$.)