EXERCISES

1. Discuss the relationships (e.g., perpendicular or parallel) to the x-axis and y-axis of the disks in examples 2.4 and 2.5. Explain how this relationship enables you to correctly determine the variable of integration.

2. The methods of disks and washers were developed separately in the text, but each is a special case of the general volume formula. Discuss the advantages of learning separate formulas versus deriving each example separately from the general formula. For example, would you prefer to learn the extra formulas or have to work each problem from basic principles? How many formulas is too many to learn?

3. To find the area of a triangle of the form Δ in section 5.1, explain why you would use y-integration. In this section, would it be easier to compute the volume of this triangle revolved about the x-axis or y-axis? Explain your preference.

4. In part (a) of example 2.7, Figure 5.23a extends from $x = -\sqrt{4-y}$ to $x = \sqrt{4-y}$ but we used $\sqrt{4-y}$ as the radius. Explain why this is the correct radius and not $2\sqrt{4-y}$.

In exercises 5–8, find the volume of the solid with cross-sectional area $A(x)$.

5. $A(x) = x + 2, -1 \le x \le 3$

6. $A(x) = 10e^{0.01x}, 0 \le x \le 10$

7. $A(x) = \pi(4-x)^2, 0 \le x \le 2$

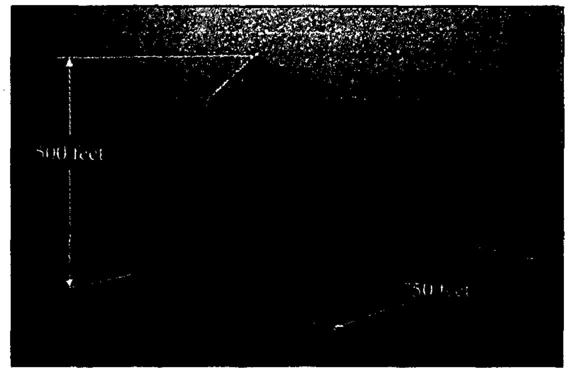
8. $A(x) = 2(x+1)^2, 1 \le x \le 4$

In exercises 9–16, set up an integral and compute the volume.

9. A swimming pool has outline $y = \pm\left(\frac{x^2}{\sqrt{x^4+1}} + 1\right)$ for $-3 \le x \le 3$ (assume units of feet for all dimensions). The depth of the pool is $6 + 3\sin\left(\frac{\pi}{6}x\right)$. Sketch a picture of the pool and compute its volume.

10. A swimming pool with the same outline as in exercise 9 has a depth of 3 feet for $-3 \le x \le -1$, a depth of 9 feet for $1 \le x \le 3$ and a depth that increases linearly from 3 to 9 for $-1 \le x \le 1$. Sketch a picture of the pool and compute its volume.

11. The great pyramid at Gizeh is 500 feet high rising from a square base of side 750 feet. Compute its volume using integration. Does your answer agree with the geometric formula?



12. Suppose that instead of completing a pyramid, the builders at Gizeh had stopped at height 250 feet (with a square plateau top of side 375 feet). Compute the volume of this structure. Explain why the volume is greater than half the volume of the pyramid in exercise 11.

13. A church steeple is 30 feet tall with square cross sections. The square at the base has side 3 feet, the square at the top has side 6 inches, and the side varies linearly in between. Compute the volume.

14. A house attic has rectangular cross sections parallel to the ground and triangular cross sections perpendicular to the ground. The rectangle is 30 feet by 60 feet at the bottom of the attic and the triangles have base 30 feet and height 10 feet. Compute the volume of the attic.

15. A pottery jar has circular cross sections of radius $4 + \sin\frac{x}{2}$ inches for $0 \le x \le 2\pi$. Sketch a picture of the jar and compute its volume.

16. A pottery jar has circular cross sections of radius $4 - \sin\frac{x}{2}$ inches for $0 \le x \le 2\pi$. Sketch a picture of the jar and compute its volume.

17. Suppose an MRI scan indicates that cross-sectional areas of adjacent slices of a tumor are as given in the table. Use Simpson's Rule to estimate the volume.

x (cm)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$A(x)$ (cm^2)	0.0	0.1	0.2	0.4	0.6	0.4	0.3	0.2	0.2	0.1	0.0

18. Suppose an MRI scan indicates that cross-sectional areas of adjacent slices of a tumor are as given in the table. Use Simpson's Rule to estimate the volume.

x (cm)	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$A(x)$ (cm^2)	0.0	0.2	0.3	0.2	0.4	0.2	0.0

19. Estimate the volume from the cross-sectional areas.

x (ft)	0.0	0.5	1.0	1.5	2.0
$A(x)$ (ft^2)	1.0	1.2	1.4	1.3	1.2

20. Estimate the volume from the cross-sectional areas.

x (m)	0.0	0.1	0.2	0.3	0.4
$A(x)$ (m^2)	2.0	1.8	1.7	1.6	1.8

x (m)	0.5	0.6	0.7	0.8
$A(x)$ (m^2)	2.0	2.1	2.2	2.4

In exercises 21–28, compute the volume of the solid formed by revolving the given region about the given line.

21. Region bounded by $y = 2 - x$, $y = 0$ and $x = 0$ about (a) the x-axis; (b) $y = 3$

22. Region bounded by $y = x^2$, $y = 0$ and $x = 2$ about (a) the x-axis; (b) $y = 4$

23. Region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ about (a) the y-axis; (b) $x = 4$

24. Region bounded by $y = 2x$, $y = 2$ and $x = 0$ about (a) the y-axis; (b) $x = 1$

25. Region bounded by $y = e^x$, $x = 0$, $x = 2$ and $y = 0$ about (a) the y-axis; (b) $y = -2$. Estimate numerically.

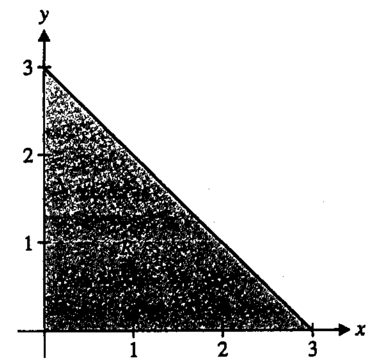
26. Region bounded by $y = \cos x$, $x = 0$ and $y = 0$ about (a) $y = 1$; (b) $y = -1$

27. Region bounded by $y = x^3$, $y = 0$ and $x = 1$ about (a) the y-axis; (b) the x-axis

28. Region bounded by $y = x^3$, $y = 0$ and $x = 1$ about (a) $x = 1$; (b) $y = -1$

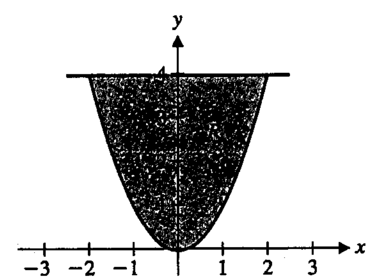
29. Let R be the region bounded by $y = 3 - x$, the x-axis and the y-axis. Compute the volume of the solid formed by revolving R about the given line.

(a) y-axis (b) x-axis (c) $y = 3$
(d) $y = -3$ (e) $x = 3$ (f) $x = -3$



30. Let R be the region bounded by $y = x^2$ and $y = 4$. Compute the volume of the solid formed by revolving R about the given line.

(a) $y = 4$ (b) y-axis
(c) $y = 6$ (d) $y = -2$



31. Let R be the region bounded by $y = x^2$, $y = 0$ and $x = 1$. Compute the volume of the solid formed by revolving R about the given line.

(a) y-axis (b) x-axis (c) $x = 1$
(d) $y = 1$ (e) $x = -1$ (f) $y = -1$

32. Let R be the region bounded by $y = x$, $y = -x$ and $x = 1$. Compute the volume of the solid formed by revolving R about the given line.

(a) x-axis (b) y-axis
(c) $y = 1$ (d) $y = -1$

33. Let R be the region bounded by $y = ax^2$, $y = h$ and the y-axis (where a and h are positive constants). Compute the volume of the solid formed by revolving this region about the y-axis. Show that your answer equals half the volume of a cylinder of height h and radius $\sqrt{h/a}$. Sketch a picture to illustrate this.

34. Use the result of exercise 33 to immediately write down the volume of the solid formed by revolving the region bounded by $y = ax^2$, $x = \sqrt{h/a}$ and the x-axis about the y-axis.

35. Suppose that the square with $-1 \le x \le 1$ and $-1 \le y \le 1$ is revolved about the y-axis. Show that the volume of the resulting solid is 2π.

36. Suppose that the circle $x^2 + y^2 = 1$ is revolved about the y-axis. Show that the volume of the resulting solid is $\frac{4}{3}\pi$.

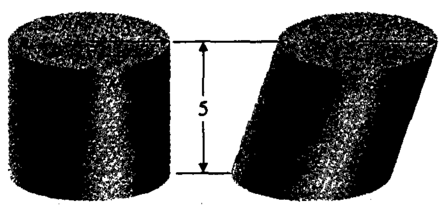
37. Suppose that the triangle with vertices $(-1, -1)$, $(0, 1)$ and $(1, -1)$ is revolved about the y-axis. Show that the volume of the resulting solid is $\frac{2}{3}\pi$.

38. Sketch the square, circle and triangle of exercises 35–37 on the same axes. Show that the relative volumes of the revolved regions (cylinder, sphere and cone, respectively) are 3:2:1.

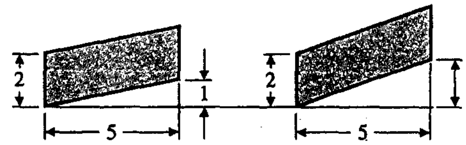
39. Verify the formula for the volume of a sphere by revolving the circle $x^2 + y^2 = r^2$ about the y-axis.

40. Verify the formula for the volume of a cone by revolving the line segment $y = -\frac{h}{r}x + h$, $0 \le x \le r$, about the y-axis.

41. Let A be a right circular cylinder with radius 3 and height 5. Let B be the tilted circular cylinder with radius 3 and height 5. Determine whether A and B enclose the same volume.



42. Determine whether or not the two indicated parallelograms have the same area.



43. Generalize the result of exercise 38 to any rectangle. That is, sketch the rectangle with $-a \le x \le a$ and $-b \le y \le b$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the triangle with vertices $(-a, -b)$, $(0, b)$ and $(a, -b)$. Show that the relative volumes of the solid formed by revolving these regions about the y-axis are 3:2:1.

44. Take the circle $(x - 2)^2 + y^2 = 1$ and revolve it about the y-axis. The resulting donut-shaped solid is called a **torus.** Compute its volume. Show that the volume equals the area of the circle times the distance travelled by the center of the circle. This is an example of **Pappus' Theorem,** dating from the fourth century B.C. Verify that the result also holds for the triangle in exercise 32, parts (c) and (d).

5.3 VOLUMES BY CYLINDRICAL SHELLS

In this section, we present an alternative to the method of washers discussed in section 5.2. This method will help with solving some problems such as example 2.7, part (d), where the method of washers led to a rather awkward integral. Beyond learning the specifics of the method, it is most important that you learn how the geometry of a given problem dictates the method of solution.

As an introduction, we return to part (d) of example 2.7. There, we let R be the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$ (see Figure 5.27a on the following page). As shown in the graph, R extends from $x = -2$ to $x = 2$. If R is revolved about the line $x = 3$, as indicated in Figure 5.27a, how would you compute the volume of the resulting solid (shown in Figure 5.27b on the following page)?

The geometry of the region R makes it awkward to integrate with respect to y, since the left-hand and right-hand boundaries of R are the left and right halves of the parabola, respectively. On the other hand, since R is nicely defined on top by $y = 4 - x^2$ and on bottom by $y = 0$, it might be much easier to integrate with respect to x. Unfortunately, in this case, the method of washers requires the y-integration. The solution lies in an alternative method of computing volumes that uses the opposite variable of integration.