We close this section with a summary of strategies for computing volumes of solids of revolution.

VOLUME OF A SOLID OF REVOLUTION

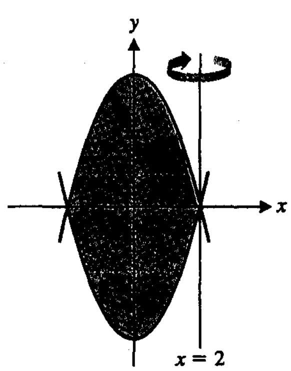
- Sketch the region to be revolved.
- Determine the variable of integration (x if the region has a well-defined top and bottom, y if the region has well-defined left and right boundaries).
- Based on the axis of revolution and the variable of integration, determine the method (disks or washers for x-integration about a horizontal axis or y-integration about a vertical axis, shells for x-integration about a vertical axis or y-integration about a horizontal axis).
- Label your picture with the inner and outer radii for disks or washers; label the radius and height for cylindrical shells.
- Set up the integral(s) and evaluate.

EXERCISES 5.3

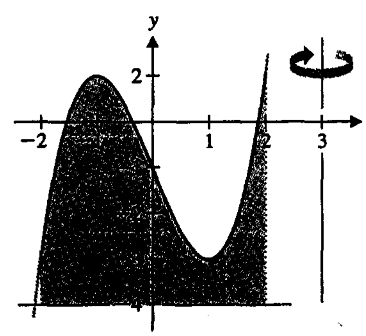
1. Explain why the method of cylindrical shells produces an integral with x as the variable of integration when revolving about a vertical axis. (Describe where the shells are and which direction to move in to go from shell to shell.)

2. Explain why the method of cylindrical shells has the same form whether or not the solid has a hole or cavity. That is, there is no need for separate methods analogous to disks and washers.

3. Suppose that the region bounded by $y = x^2 - 4$ and $y = 4 - x^2$ is revolved about the line $x = 2$. Carefully explain which method (disks, washers or shells) would be easiest to use to compute the volume.



4. Suppose that the region bounded by $y = x^3 - 3x - 1$ and $y = -4$, $-2 \le x \le 2$, is revolved about $x = 3$. Explain what would be necessary to compute the volume using the method of washers, and what would be necessary to use the method of cylindrical shells. Which method would you prefer and why?



In exercises 5–12, sketch the region, draw in a typical shell, identify the radius and height of each shell and compute the volume.

5. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = 2$

6. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = -2$

7. The region bounded by $y = x, y = -x$ and $x = 1$ revolved about the y-axis

8. The region bounded by $y = x, y = -x$ and $x = 1$ revolved about $x = 1$

9. The region bounded by $y = x, y = -x$ and $y = 1$ revolved about $y = 2$

10. The region bounded by $y = x, y = -x$ and $y = 1$ revolved about $y = -2$

11. The right half of $x^2 + (y-1)^2 = 1$ revolved about the x-axis

12. The right half of $x^2 + (y-1)^2 = 1$ revolved about $y = 2$

In exercises 13–20, use cylindrical shells to compute the volume.

13. The region bounded by $y = x^2$ and $y = 2 - x^2$, revolved about $x = -2$

14. The region bounded by $y = x^2$ and $y = 2 - x^2$, revolved about $x = 2$

15. The region bounded by $x = y^2$ and $x = 1$ revolved about $y = -2$

16. The region bounded by $x = y^2$ and $x = 1$ revolved about $y = 2$

17. The region bounded by $y = x$ and $y = x^2 - 2$ revolved about $x = 2$

18. The region bounded by $y = x$ and $y = x^2 - 2$ revolved about $x = 3$

19. The region bounded by $x = (y-1)^2$ and $x = 1$ revolved about the x-axis

20. The region bounded by $x = (y-1)^2$ and $x = 1$ revolved about $y = 2$

In exercises 21–30, use the best method available to find each volume.

21. The region bounded by $y = 4 - x, y = 4$ and $y = x$ revolved about

(a) x-axis (b) y-axis (c) $x = 4$ (d) $y = 4$

22. The region bounded by $y = x + 2, y = -x - 2$ and $x = 0$ revolved about

(a) $y = -2$ (b) $x = -2$ (c) y-axis (d) x-axis

23. The region bounded by $y = x$ and $y = x^2 - 6$ revolved about

(a) $x = 3$ (b) $y = 3$ (c) $x = -3$ (d) $y = -6$

24. The region bounded by $x = y^2$ and $x = 2 + y$ revolved about

(a) $x = -1$ (b) $y = -1$ (c) $x = -2$ (d) $y = -2$

25. The region bounded by $y = \cos x$ and $y = x^4$ revolved about

(a) $x = 2$ (b) $y = 2$ (c) x-axis (d) y-axis

26. The region bounded by $y = \sin x$ and $y = x^2$ revolved about

(a) $y = 1$ (b) $x = 1$ (c) y-axis (d) x-axis

27. The region bounded by $y = x^2$, $y = 2 - x$ and $x = 0$ revolved about

(a) x-axis (b) y-axis (c) $x = 1$ (d) $y = 2$

28. The region bounded by $y = 2 - x^2, y = x\ (x > 0)$ and the y-axis revolved about

(a) x-axis (b) y-axis (c) $x = -1$ (d) $y = -1$

29. The region bounded by $y = 2 - x, y = x - 2$ and $x = y^2$ revolved about

(a) x-axis (b) y-axis

30. The region bounded by $y = e^x - 1, y = 2 - x$ and the x-axis revolved about

(a) x-axis (b) y-axis

In exercises 31–36, the integral represents the volume of a solid. Sketch the region and axis of revolution that produce the solid.

31. $\int_0^2 \pi(2x - x^2)^2\, dx$

32. $\int_{-2}^2 \pi[(4 - x^2 + 4)^2 - (x^2 - 4 + 4)^2]\, dx$

33. $\int_0^1 \pi[(\sqrt{y})^2 - y^2]\, dy$

34. $\int_0^2 \pi(4 - y^2)^2\, dy$

35. $\int_0^1 2\pi x(x - x^2)\, dx$

36. $\int_0^2 2\pi(4 - y)(y + y)\, dy$

37. Use a method similar to our derivation of equation (3.1) to derive the following fact about a circle of radius R. Area $= \pi R^2 = \int_0^R c(r)\, dr$, where $c(r) = 2\pi r$ is the circumference of a circle of radius r.

38. You have probably noticed that the circumference of a circle $(2\pi r)$ equals the derivative with respect to r of the area of the circle (πr^2). Use exercise 37 to explain why this is not a coincidence.

39. A jewelry bead is formed by drilling a $\frac{1}{2}$-cm hole from the center of a 1-cm sphere. Explain why the volume is given by

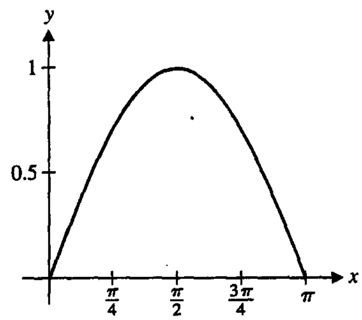
$\int_{1/2}^{1} 4\pi x\sqrt{1-x^2}\,dx$. Evaluate this integral or compute the volume in some easier way.

40. Find the size of the hole in exercise 39 such that exactly half the volume is removed.

41. An anthill is in the shape formed by revolving the region bounded by $y = 1 - x^2$ and the x-axis about the y-axis. A researcher removes a cylindrical core from the center of the hill. What should the radius be to give the researcher 10% of the dirt?

42. From a sphere of radius R, a hole of radius r is drilled out of the center. Compute the volume removed in terms of R and r. Compute the length L of the hole in terms of R and r. Rewrite the volume in terms of L. Is it reasonable to say that the volume removed depends on L and not on R?

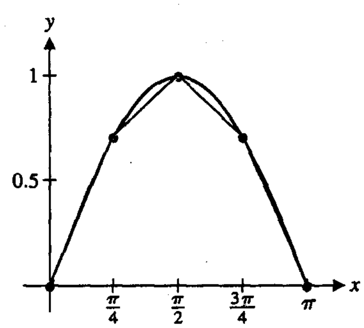
5.4 ARC LENGTH AND SURFACE AREA



$y = \sin x$.

In this section, we use the definite integral to investigate two additional measures of geometric size. Length and area are quantities you already understand intuitively. But, as you have learned with area, the calculation of these quantities can be surprisingly challenging for many geometric shapes. The calculations that we explore here are complicated by an increase in the dimension. Specifically, we will compute the length (a one-dimensional measure) of a curve in two dimensions and we will compute the area (a two-dimensional measure) of a surface in three dimensions. As always, pay particular attention to the derivations. As we have done a number of times now, we start with an approximation and then proceed to the exact solution, using the notion of limit.

Arc Length



Four line segments approximating $y = \sin x$.

What could we mean by the *length* of the portion of the sine curve shown in Figure 5.34a? (We call the length of a curve its **arc length.**) If the curve represented a road, you could measure the length on your car's odometer by driving along that section of road. If the curve were actually a piece of string, you could straighten out the string and then measure its length with a ruler. Both of these ideas are very helpful intuitively. They both involve turning the problem of measuring length in two dimensions into the (much easier) problem of measuring the length in one dimension.

To accomplish this mathematically, we first approximate the curve with several line segments joined together. In Figure 5.34b, the line segments connect the points (0, 0), $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{\pi}{2}, 1\right)$, $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $(\pi, 0)$ on the curve $y = \sin x$. An approximation of the arc length s of the curve is given by the sum of the lengths of the line segments:

$$s \approx \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2}$$
$$+ \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}} - 1\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \approx 3.79.$$

8	3.8125
16	3.8183
32	3.8197
64	3.8201
128	3.8201

You might notice that this estimate is too small. (Why is that?) We could improve our approximation by using more than four line segments. In the table at left, we show estimates