Solutions

Name:

Math 107H Exam 1

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Find each of the following integrals.

Note that " $\int_3^x f(t)\,dt + C$ " is not a sufficient computation of an antiderivative!

Some formulas of potential use can be found at the bottom of the last page of the exam.

1. (10 pts.) $\int (x+2)^{3/2}\,dx$

$u = x+2 \qquad du = dx$

$$\int (x+2)^{3/2}\,dx = \int u^{3/2}\,du \Big|_{u=x+2} = \frac{2}{5}u^{\frac{5}{2}} + C \Big|_{u=x+2}$$

$$\boxed{= \frac{2}{5}(x+2)^{5/2} + C}$$

3 is odd!

2. (15 pts.) $\int_0^{\pi/2} \sin^3 x \, dx$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \, (\sin x \, dx)$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\,(\sin x \, dx)$$

$u = \cos x \quad du = -\sin x \, dx$

$x = 0 \rightsquigarrow u = 1$

$x = \frac{\pi}{2} \rightsquigarrow u = 0$

$$= \int_1^0 (1-u^2)\,(-du) = \int_1^0 u^2 - 1 \, du$$

$$= \frac{u^3}{3} - u \Big|_1^0 = \boxed{(0-0) - \left(\frac{1}{3} - 1\right)}$$

$$= 0 - \left(-\frac{2}{3}\right) = \frac{2}{3}$$

3. (10 pts.) $\int \frac{x^2+x-3}{x^{1/2}}\, dx$ $= \int \frac{x^2}{x^{1/2}} + \frac{x}{x^{1/2}} - \frac{3}{x^{1/2}}\, dx$

$= \int x^{3/2} + x^{1/2} - 3x^{-1/2}\, dx$

$$\boxed{= \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} - 3(2x^{1/2}) + C}$$

4. (15 pts.) $\int_0^1 e^{\sqrt{x}}\,dx$

$u = \sqrt{x}$ $\;\; x^{\frac{1}{2}}$ $\quad du = \frac{1}{2} x^{-\frac{1}{2}} dx = \frac{1}{2\sqrt{x}} dx$

$u = 0 \rightarrow x = 0$

$u = 1 \sim x = 1$

$$= \int_0^1 (2\sqrt{x}) e^{\sqrt{x}} \left(\frac{dx}{2\sqrt{x}}\right) = \int_0^1 2u e^u \, du$$

$$= 2\int_0^1 u e^u \, du$$

$v = u \quad dw = e^u du$

$dv = du \quad w = e^u$

$$= 2\left(u e^u \Big|_0^1 - \int_0^1 e^u \, du \right)$$

$$= 2\left(u e^u \Big|_0^1 - e^u \Big|_0^1 \right)$$

$$\boxed{= 2\left((1 \cdot e - 0 \cdot 1) - (e - 1) \right)}$$

$$= 2(e - 0 - e + 1) = 2(1) = \underline{\underline{2}}.$$

$u = e^{\sqrt{x}}$ also works! Gets you to $2\int \ln u \, du \Big|_{u = e^{\sqrt{x}}} \ldots$

5. (15 pts.) $\int \frac{dx}{(x+1)^2(x+4)} = (*)$

$$\frac{1}{(x+1)^2(x+4)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+4)}$$

$$= \frac{A(x+1)(x+4) + B(x+4) + C(x+1)^2}{(x+1)^2(x+4)}$$

So need: $1 = A(x+1)(x+4) + B(x+4) + C(x+1)^2$

Set $x=-1$: $1 = A(0)(3) + B(3) + C(0)^2 = 3B$

$\Rightarrow B = 1/3$

Set $x=-4$: $1 = A(-3)(0) + B(0) + C(-3)^2 = 9C$

$\Rightarrow C = 1/9$

Set $x=0$: $1 = A(1)(4) + \frac{1}{3}(4) + \frac{1}{9}(1)^2 = 4A + \frac{4}{3} + \frac{1}{9}$

$\Rightarrow 4A = 1 - \frac{4}{3} - \frac{1}{9} = \frac{9-12-1}{9} = \frac{-4}{9} \Rightarrow A = -\frac{1}{9}$

So $(*) = \int -\frac{1}{9}\frac{1}{x+1} + \frac{1}{3}\frac{1}{(x+1)^2} + \frac{1}{9}\frac{1}{x+4}\,dx$

(u=x+1) (du=dx   (u=x+1) (du=dx)   (u=x+4) (du=dx)

$$\boxed{= -\frac{1}{9}\ln|x+1| + \frac{1}{3}\left(\frac{-1}{(x+1)}\right) + \frac{1}{9}\ln|x+4| + C}$$

6. (15 pts.) $\boxed{\int e^{-x}\sin(3x)\,dx}$

(!)

$u = \sin(3x) \quad dv = e^{-x}dx$

$du = 3\cos(3x)dx \quad v = -e^{-x}$

$= -e^{-x}\sin 3x - \int -3e^{-x}\cos(3x)\,dx$

$= -e^{-x}\sin 3x + 3\int e^{-x}\cos(3x)\,dx$

$u = \cos(3x) \quad dv = e^{-x}dx$

$du = -3\sin(3x) \quad v = -e^{-x}$

$= -e^{-x}\sin 3x + 3\left(-e^{-x}\cos(3x) - \int -(-3)e^{-x}\sin 3x\,dx\right)$

$= -e^{-x}\sin 3x - 3e^{-x}\cos 3x - 9\boxed{\int e^{-x}\sin 3x\,dx}$

So $10\int e^{-x}\sin 3x\,dx = -e^{-x}\sin 3x - 3e^{-x}\cos 3x$

So: $\boxed{\int e^{-x}\sin 3x\,dx = -\frac{1}{10}e^{-x}\sin 3x - \frac{3}{10}e^{-x}\cos 3x + C}$

7. (20 pts.) $\int (x^2+1)^{3/2}\, dx = \int \left(\sqrt{x^2+1}\,\right)^3 dx$

Think: $\tan^2 u + 1 = \sec^2 u$

$x = \tan u \quad dx = \sec^2 u\, du$

$\cancel{dx}\quad x^2+1 = \tan^2 u + 1 = \sec^2 u$

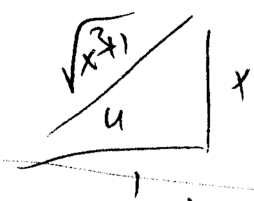
$\sqrt{x^2+1} = \sec u$

$$= \int (\sec u)^3 (\sec^2 u\, du)\Big|_{x=\tan u}$$

$$= \int \sec^5 u\, du\Big|_{x=\tan u} \overset{(n=5)}{=} \frac{1}{4}\sec^3 u \tan u\Big|_{x=\tan u} + \frac{3}{4}\int \sec^3 u\, du\Big|_{x=\tan u}$$

$$\overset{(n=3)}{=} \frac{1}{4}\sec^3 u \tan u + \frac{3}{4}\left(\frac{1}{2}\sec u \tan u + \frac{1}{2}\int \sec u\, du\right)\Big|_{x=\tan u}$$

$$= \frac{1}{4}\sec^3 u \tan u + \frac{3}{8}\sec u \tan u + \frac{3}{8}\ln|\sec u + \tan u| + C\Big|_{x=\tan u}$$



$\tan u = x$

$\sec u = \sqrt{x^2+1}$

$$\boxed{\begin{aligned} = \frac{1}{4}\left(\sqrt{x^2+1}\right)^3 x + \frac{3}{8}\left(\sqrt{x^2+1}\right)x \\ + \frac{3}{8}\ln\left|\sqrt{x^2+1} + x\right| + C \end{aligned}}$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

$$c^2 \int \frac{dy}{(y^2+c^2)^k} = \frac{1}{(2k-2)}\cdot\frac{y}{(y^2+c^2)^{k-1}} + \frac{(2k-3)}{(2k-2)}\int \frac{dy}{(y^2+c^2)^{k-1}}$$