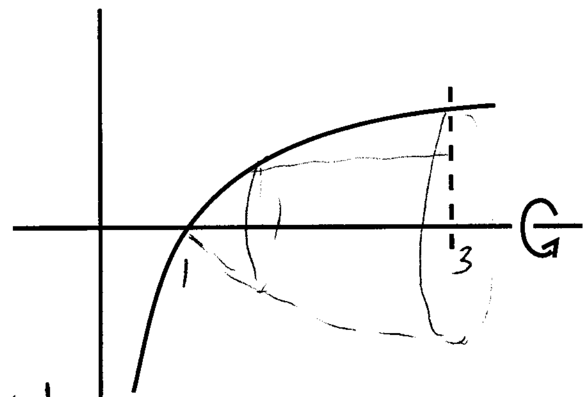
Solutions

Name:

Math 107H Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

1. (20 pts.) Find the volume of the region obtained by revolving the region under the graph of $f(x) = \ln x$ from $x = 1$ to $x = 3$ around the x-axis (see figure).



By slices dx:

Volume $= \int_1^3 \pi (\ln x)^2 \, dx$

$u = (\ln x)^2 \quad dv = dx$

$\int (\ln x)^2 dx \quad du = \frac{2\ln x}{x} dx \quad v = x$

$u = \ln x \quad dv = dx$

$= x(\ln x)^2 - \int 2\ln x \, dx \quad du = \frac{1}{x} dx \quad v = x$

$= x(\ln x)^2 - 2\left(x\ln x - \int dx\right) = x(\ln x)^2 - 2x\ln x + 2x + C$

So Volume $= \pi\left(x(\ln x)^2 - 2x\ln x + 2x\right)\Big|_1^3 = \boxed{\pi\Big(\left(3(\ln 3)^2 - 6\ln 3 + 6\right) - \left((\ln 1)^2 + 2\ln 1 + 2\right)\Big)}$

$= \pi\left(3(\ln 3)^2 - 6\ln 3 + 4\right)$

— or —

Shells dy $\quad y = \ln x, \; x = e^y \quad x = 1 \to y = 0$, $\; x = 3 \to y = \ln 3$

Volume $= \int_0^{\ln 3} 2\pi y (3 - e^y) \, dy = 6\pi \int_0^{\ln 3} y \, dy - 2\pi \int_0^{\ln 3} y e^y \, dy \quad u = y \;\; dv = e^y dy \;\; du = dy \;\; v = e^y$

$= 6\pi \left(\frac{y^2}{2}\right)\Big|_0^{\ln 3} - 2\pi\left(y e^y \Big|_0^{\ln 3} - \int_0^{\ln 3} e^y \, dy\right)$

$= 3\pi y^2 \Big|_0^{\ln 3} - 2\pi y e^y \Big|_0^{\ln 3} + 2\pi e^y \Big|_0^{\ln 3} = \left(3\pi y^2 - 2\pi y e^y + 2\pi e^y\right)\Big|_0^{\ln 3}$

$\boxed{= \left(3\pi(\ln 3)^2 - 2\pi(\ln 3)(3) + 2\pi(3)\right) - (0 - 0 + 2\pi)} = 3\pi(\ln 3)^2 - 6\pi(\ln 3) + 4\pi$

2. (15 pts.) Set up, **but do not evaluate**, the integral which will compute the arclength of the graph of the function $g(x) = \ln(x^2 - 1)$ from $x = 2$ to $x = 4$.

$$g'(x) = \frac{1}{x^2-1}(2x) = \frac{2x}{x^2-1}$$

$$1 + (g'(x))^2 = 1 + \left(\frac{2x}{x^2-1}\right)^2$$

$$\text{Arclength} = \boxed{\int_2^4 \sqrt{1 + \left(\frac{2x}{x^2-1}\right)^2}\, dx}$$

Note! We can evaluate this integral!

$$1 + \left(\frac{2x}{x^2-1}\right)^2 = \frac{(x^2-1)^2 + (2x)^2}{(x^2-1)^2} = \frac{x^4 - 2x^2 + 1 + 4x^2}{(x^2-1)^2}$$

$$= \frac{x^4 + 2x^2 + 1}{(x^2-1)^2} = \frac{(x^2+1)^2}{(x^2-1)^2} = \left(\frac{x^2+1}{x^2-1}\right)^2 = \left(1 + \frac{2}{x^2-1}\right)^2$$

$$\text{So Arclength} = \int_2^4 \sqrt{\left(1 + \frac{2}{x^2-1}\right)^2}\, dx = \int_2^4 1 + \frac{2}{x^2-1}\, dx$$

(partial fractions!)

$$= \int_2^4 1 + \frac{1}{x-1} - \frac{1}{x+1}\, dx = x + \ln|x-1| - \ln|x+1| \Big|_2^4$$

$$= (4 + \ln 3 - \ln 5) - (2 + \ln 1 - \ln 3) = 2 + \ln 3 + \ln 3 - \ln 5$$

$$= \boxed{2 + \ln\left(\frac{9}{5}\right)}$$

3. (15 pts.) Volodinaria are a form of bacteria whose population, in the presence of sufficient nutrients, will follow an exponential growth law $P'(t) = kP(t)$. [We will measure t in days.] If an initial population of 10000 can grow to 30000 in four (4) days, what is the value of the growth constant k ?

$$\frac{dy}{dt} = y' = ky \,, \quad \frac{dy}{y} = k\,dt \,,$$

$$\int \frac{dy}{y} = \int k\,dt \,, \quad \ln y = kt + C \quad y = e^{kt+C} = e^{kt}e^{C} = C e^{kt}$$

$$y(0) = Ce^{k \cdot 0} = Ce^{0} = C = 10000 \,, \quad y(t) = 10000\,e^{kt}$$

$$y(4) = 30000 = 10000\,e^{4k} \Longrightarrow \frac{30000}{10000} = 3 = e^{4k}$$

$$\Longrightarrow \ln 3 = 4k \Rightarrow \boxed{k = \frac{1}{4}\ln 3}$$

4. (10 pts. each) Find (if they exist) the limits of the following sequences:

(a) $a_n = \frac{(\ln n)^n}{n^2}$

$$a_n = \frac{(\ln n)^n}{((n^{1/n})^n)^2} = \frac{(\ln n)^n}{((n^{1/n})^2)^n} = \left(\frac{\ln n}{(n^{1/n})^2}\right)^n$$

But $n^{1/n} \to 1$ and $\ln n \to \infty$ so for large n, $a_n = (\text{large})^n = \underline{\text{really}}$ large, so $a_n \to \infty$ as $n \to \infty$.

— or —

Eventually, $\ln n > 2$ so eventually $a_n > \frac{2^n}{n^2}$. But by L'Hôpital, $\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{2^n \ln 2}{2n} = \lim_{n\to\infty} \frac{2^n (\ln 2)^2}{2} = \infty$ since $2^n \to \infty$

So $a_n > 2^n/n^2$ and $2^n/n^2 \to \infty$, so $a_n \to \infty$ as $n \to \infty$.

(b) $b_n = \frac{n - n^2 + 4}{4n^2 + 4n + 3}$ $= \frac{\frac{1}{n^2}(n - n^2 + 4)}{\frac{1}{n^2}(4n^2 + 4n + 3)} = \frac{\frac{1}{n} - 1 + \frac{4}{n^2}}{4 + \frac{4}{n} + \frac{3}{n^2}}$

dominant terms

$$\to \frac{0 - 1 + 0}{4 + 0 + 0} = \boxed{\frac{-1}{4}}$$ as $n \to \infty$, since $\frac{1}{n} \to 0$ as $n \to \infty$

On 4(a) several people wanted to use L'Hôpital, which is fine, but the derivative of $(\ln n)^n$ is more subtle than you might think: $(\ln n)^n = e^{n \ln n}$, whose derivative is

$$e^{n \ln n}\left(1 \cdot \ln n + n\left(\tfrac{1}{n}\right)\right) = (\ln n)^n (\ln n + 1) \; [!]$$

5. (10 pts. each) Determine the convergence or divergence of the following sequences:

(a) $\sum_{n=1}^{\infty} n^2\left(2+\frac{(-1)^n}{n}\right)^{-n} = \sum a_n$

Root test! $a_n^{1/n} = (n^2)^{1/n}\left(\left(2+\frac{(-1)^n}{n}\right)^{-n}\right)^{1/n}$

$= (n^{1/n})^2\left(2+\frac{(-1)^n}{n}\right)^{-1}$

But $n^{1/n} \to 1$ as $n\to\infty$ and $\frac{(-1)^n}{n} \to 0$ as $n\to\infty$

(by squeeze play: $\frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$ and both ends $\to 0$ as $n\to\infty$)

So $a_n^{1/n} \to (1)^2(2+0)^{-1} = \frac{1}{2} < 1$, so $\boxed{\sum a_n \text{ converges}}$

(b) $\sum_{n=2}^{\infty} \frac{n^{3/2}+n^{5/3}}{n^3+1} = \sum a_n$ limit compare to $b_n = \frac{n^{5/3}}{n^3} = \frac{1}{n^{4/3}}$

dominant terms

$\frac{a_n}{b_n} = \frac{n^{3/2}+n^{5/3}}{n^{5/3}} \cdot \frac{n^3}{n^3+1} = \frac{n^{-1/6}+1}{1} \cdot \frac{1}{1+n^{-3}}$

$\to \frac{0+1}{1} \cdot \frac{1}{1+0} = 1$ as $n\to\infty$ (since $\frac{1}{n^{1/6}}, \frac{1}{n^3} \to 0$)

So since $\sum b_n = \sum 1/n^{4/3}$ converges (p-series, $p=4/3>1$)

$\boxed{\sum a_n \text{ converges}}$

or $a_n < \frac{n^{3/2}+n^{5/3}}{n^3} = n^{-3/2}+n^{-4/3}$

and $\sum n^{-3/2}+n^{-4/3}$ converges by the integral test (or: it is the sum of two convergent p-series!).

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n \arctan n}{n!} = \sum a_n$ [Hint: what is $\lim_{n\to\infty} \arctan n$?]

Ratio Test! $\displaystyle \frac{a_{n+1}}{a_n} = \frac{5^{n+1}\arctan(n+1)}{(n+1)!} \cdot \frac{n!}{5^n \arctan(n)}$

$$= \frac{5^{n+1}}{5^n} \cdot \frac{\arctan(n+1)}{\arctan(n)} \cdot \frac{n!}{(n+1)!} = 5\,\frac{\arctan(n+1)}{\arctan(n)}\,\frac{1}{n+1}$$

But! as $n \to \infty$, $\arctan(n) \to \frac{\pi}{2}$

(angle whose tangent is large is close to $\frac{\pi}{2}$)

so $\arctan(n+1) \to \frac{\pi}{2}$ as well, so

$$\frac{\arctan(n+1)}{\arctan(n)} \to \frac{\pi/2}{\pi/2} = 1. \text{ so}$$

$\displaystyle \frac{a_{n+1}}{a_n} \to 5 \cdot 1 \cdot 0 = 0 < 1$ so $\boxed{\sum a_n \text{ converges}}$

by the ratio(n) test.