Math 107H Section 3, Solutions to Exam 3

1. Show that the alternating series $\sum_{n=2}^{\infty}(-1)^n \ln\left(\frac{n+1}{n}\right)$ converges.

How close to the infinite sum can we guarantee that $\sum_{n=2}^{1000}(-1)^n \ln\left(\frac{n+1}{n}\right)$ is?

[FYI: This series, in fact, sums to $\ln\left(\frac{4}{\pi}\right)$.]

$\sum_{n=2}^{\infty}(-1)^n a_n \qquad a_n = \ln\left(\frac{n+1}{n}\right) = f(n)$ for $f(x) = \ln\left(\frac{x+1}{x}\right)$

Since $n+1>n$, $\frac{n+1}{n}>1$ so $\ln\left(\frac{n+1}{n}\right) > \ln(1) = 0$, so $a_n > 0$

$f(x) = \ln\left(\frac{x+1}{x}\right) = \ln(x+1) - \ln(x)$, so

$f'(x) = \frac{1}{x+1} - \frac{1}{x} = \frac{x-(x+1)}{x(x+1)} = \frac{-1}{x(x+1)} < 0$ so a_n is decreasing.

Finally, as $n \to \infty$, $a_n = \ln\left(\frac{n+1}{n}\right) = \ln\left(1+\frac{1}{n}\right) \longrightarrow \ln(1+0) = \ln(1) = 0$

So by the alternating series test, $\sum_{n=2}^{\infty}(-1)^n a_n$ converges.

Setting $L = \sum_{n=2}^{\infty}(-1)^n a_n$, then we also know that

$$\left|L - \sum_{n=2}^{1000}(-1)^n a_n\right| \le a_{1001} = \ln\left(\frac{1002}{1001}\right)$$

So the partial sum is within $\boxed{\ln\left(\frac{1002}{1001}\right) = \ln(1002) - \ln(1001)}$ of the sum.

[Note: $\ln\left(\frac{1002}{1001}\right) = \int_{1001}^{1002} \frac{1}{x}\,dx \le \int_{1001}^{1002} \frac{1}{1001}\,dx = \frac{1}{1001}$.]

2. Use the Taylor series for $f(x) = e^x$, centered at $x = 0$, to find a power series (centered at 0) whose sum is

$$g(x) = \frac{e^x - 1}{x}.$$

Use this to compute $g^{(85)}(0)$ (that is, the 85-th derivative of g, evaluated at $x = 0$). [Note: As written, $g(x)$ is not defined at $x = 0$. By declaring $g(0) = 1$, we do make it continuous (and differentiable), as your work on this problem will show!]

we know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (its Taylor series)

so $e^x - 1 = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

so $\underbrace{\frac{e^x - 1}{x}}_{g(x)} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$ (by reindexing)

so since $g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$ (if it is a power series!)

we have $\frac{g^{(n)}(0)}{n!} = \frac{1}{(n+1)!}$, so $g^{(n)}(0) = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

so, in particular, $g^{(85)}(0) = \frac{1}{85+1} = \boxed{\frac{1}{86}}$

3. Find the Taylor polynomial $P_4(x)$ of degree 4, centered at $x = 0$, for the function

$$f(x) = x\ln(x+1)$$

Using Taylor's Theorem, give a bound on the size of the error in using $P_4(x)$ to estimate $f(x)$, when $-0.2 < x < 0.2$.

$f(x) = x\ln(x+1)$

$f'(x) = \ln(x+1) + x\frac{1}{x+1} = \ln(x+1) + \frac{x}{x+1}$

$f''(x) = \frac{1}{x+1} + \frac{(x+1)(1) - x(1)}{(x+1)^2} = (x+1)^{-1} + (x+1)^{-2}$

$f'''(x) = (-1)(x+1)^{-2} + (-2)(x+1)^{-3}$

$f^{(4)}(x) = 2(x+1)^{-3} + 6(x+1)^{-4}$

$f^{(5)}(x) = (-6)(x+1)^{-4} + (-24)(x+1)^{-5}$

$f^{(6)}(x) = 24(x+1)^{-5} + 120(x+1)^{-6}$

$f(0) = 0\ln(1) = 0$

$f'(0) = \ln(1) + \frac{0}{1} = 0$

$f''(0) = 1 + 1 = 2$

$f'''(0) = (-1) + (-2) = -3$

$f''''(0) = 2 + 6 = 8$

So $P_4(x) = 0 + 0\cdot x + \frac{2}{2!}x^2 + \frac{(-3)}{3!}x^3 + \frac{8}{4!}x^4$

$$\boxed{= x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4}$$

By Taylor $|f(x) - P_4(x)| \le \frac{M}{5!}|x-0|^5 \le \frac{M}{5!}(.2)^5$ where $M = \max$ of $|f^{(5)}(x)|$ on $-.2 \le x \le .2$. But since $|f^{(5)}(x)| = 6(x+1)^{-4} + 24(x+1)^{-5}$ has $(|f^{(5)}(x)|)' = -24(x+4)^{-5} + 120(x+4)^{-6} \le 0$ on this interval so $|f^{(5)}(x)|$ is decreasing, so achieves its maximum at the left endpt $x = -.2$, so

$$\boxed{|f(x) - P_4(x)| \le \frac{(.2)^5}{5!}\left(6(.8)^{-4} + 24(.8)^{-5}\right) \text{ for } |x| < 0.2 \approx 2.34375 \times 10^{-4}}$$

4. For the polar curve

$$r = 1 + 2\sin\theta, \quad = f(\theta)$$

find the values of θ, $0 \le \theta \le 2\pi$ where the curve has a *horizontal* tangent line. [You may leave your answers in a "pure" form, as values of the functions $\arctan x$, $\arcsin x$, etc.]

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left((1+2\sin\theta)\sin\theta\right)}{\frac{d}{d\theta}\left((1+2\sin\theta)\cos\theta\right)}$$

horizontal tangent!

$$= \frac{(\sin\theta + 2\sin^2\theta)'}{(\cos\theta + 2\sin\theta\cos\theta)'} = \frac{\cos\theta + 4\sin\theta\cos\theta}{-\sin\theta + 2\cos^2\theta - 2\sin^2\theta} = 0$$

provided $\cos\theta + 4\sin\theta\cos\theta = \cos\theta(1+4\sin\theta) = 0$

so either $\cos\theta = 0$ (so $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$)

or $1 + 4\sin\theta = 0$ (so $\sin\theta = -1/4$, (and $0 \le \theta \le 2\pi$))

$\theta = \cancel{\arcsin(-1/4)}$

so $\theta = \arcsin(\frac{1}{4}) + \pi$ or $\theta = \arcsin(-\frac{1}{4}) + 2\pi$

$= 2\pi - \arcsin(1/4)$

so $\boxed{\theta = \frac{\pi}{2}, \arcsin(\frac{1}{4}) + \pi, \frac{3\pi}{2}, \text{ and } 2\pi - \arcsin(\frac{1}{4})}$

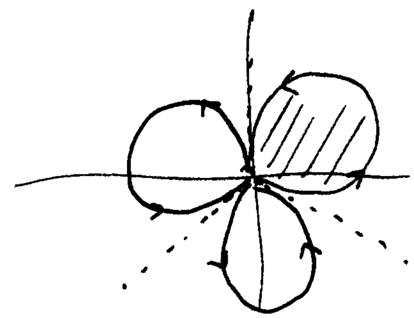
5. Find the area inside one petal of the 3-petaled rose, given by the polar equation

$$r = 1 + \sin(3\theta)$$

[One petal is defined by consecutive values of θ for which $r = 0$; you should find such a pair as part of your solution.]

$r = 1 + \sin(3\theta) = 0$ for $\sin(3\theta) = -1$ so

$3\theta = \frac{-\pi}{2}, \frac{3\pi}{2}$, so $\boxed{\theta = \frac{-\pi}{6}, \frac{\pi}{2}}$ (other choices work, as well)



so Area $= \int_{-\pi/6}^{\pi/2} \frac{1}{2}(1+\sin(3\theta))^2 \, d\theta$

$= \frac{1}{2}\int_{-\pi/6}^{\pi/2} 1 + 2\sin(3\theta) + \sin^2(3\theta) \, d\theta$

$= \frac{1}{2}\int_{-\pi/6}^{\pi/2} 1 + 2\sin(3\theta) + \frac{1}{2}(1 - \cos(6\theta)) \, d\theta$

$= \frac{1}{2}\int_{-\pi/6}^{\pi/2} \frac{3}{2} + 2\sin(3\theta) - \frac{1}{2}\cos(6\theta) \, d\theta$

$= \frac{1}{2}\left(\frac{3}{2}\theta - \frac{2}{3}\cos(3\theta) - \frac{1}{12}\sin(6\theta)\right)\Big|_{-\pi/6}^{\pi/2}$

$= \frac{1}{2}\left(\left(\frac{3}{2}\frac{\pi}{2} - \frac{2}{3}\cos\left(\frac{3\pi}{2}\right) - \frac{1}{12}\sin(3\pi)\right) - \left(\frac{3}{2}\left(\frac{-\pi}{6}\right) - \frac{2}{3}\cos\left(\frac{-\pi}{2}\right) - \frac{1}{12}\sin(-\pi)\right)\right)$

$= \frac{1}{2}\left(\left(\frac{3\pi}{4} - 0 - 0\right) - \left(-\frac{\pi}{4} - 0 - 0\right)\right)$

$= \frac{1}{2}\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) = \frac{1}{2}(\pi) = \boxed{\frac{\pi}{2}}$

6. Find the length of the polar curve

$$r = 1 - \theta^2 = f(\theta) \qquad f'(\theta) = -2\theta$$

from $\theta = -1$ to $\theta = 1$.

$$\text{Arclength} = \int_{-1}^{1} \left((f(\theta))^2 + (f'(\theta))^2 \right)^{1/2} d\theta$$

$$= \int_{-1}^{1} \left((1-\theta^2)^2 + (-2\theta)^2 \right)^{1/2} d\theta$$

$$= \int_{-1}^{1} \left(1 - 2\theta^2 + \theta^4 + 4\theta^2 \right)^{1/2} d\theta$$

$$= \int_{-1}^{1} \left(1 + 2\theta^2 + \theta^4 \right)^{1/2} d\theta = \int_{-1}^{1} \left((1+\theta^2)^2 \right)^{1/2} d\theta$$

$$= \int_{-1}^{1} |1+\theta^2| \, d\theta = \int_{-1}^{1} 1 + \theta^2 \, d\theta = \theta + \frac{\theta^3}{3} \Big|_{-1}^{1}$$

$$= \left(\left(1 + \frac{1}{3}\right) - \left(-1 + \frac{-1}{3}\right) \right) = \frac{4}{3} - \left(\frac{-4}{3}\right) = \boxed{\frac{8}{3}}$$