Name: Solutions

Math 107H, Section 3

Quiz number 1

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Find the following antiderivatives.

1. $\int 3x^{3/2} - 7x^{-2} + 4\ dx$

$= 3\int x^{3/2}dx - 7\int x^{-2}dx + 4\int 1\,dx$

$= \boxed{3\left(\frac{x^{5/2}}{5/2}\right) - 7\left(\frac{x^{-1}}{-1}\right) + 4x + C}$

$= \frac{6}{5}x^{5/2} + 7x^{-1} + 4x + C$

2. $\int \frac{5\sin x - 3\cos x}{2}\ dx = \frac{5}{2}\int \sin x\,dx - \frac{3}{2}\int \cos x\,dx$

$= \boxed{\frac{5}{2}(-\cos x) - \frac{3}{2}(\sin x) + C}$

$= -\frac{5}{2}\cos x - \frac{3}{2}\sin x + C.$

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Quiz number 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Find the following integrals.

1. $\int_1^2 (1-2x)^{4/3}\, dx$

$u = 1-2x \qquad du = -2dx \qquad x=1 \quad u=-1 \qquad x=2 \quad u=-3$

$$= \int_1^2 \frac{(1-2x)^{4/3}}{-2}(-2dx) = -\frac{1}{2}\int_{-1}^{-3} u^{4/3}\, du$$

$$= -\frac{1}{2}\left.\frac{u^{7/3}}{7/3}\right|_{-1}^{-3} = \boxed{\left(-\frac{1}{2}\right)\left(\frac{3}{7}\right)\left((-3)^{7/3} - (-1)^{7/3}\right)}$$

2. $\int \frac{1}{\sqrt{x}+5}\, dx$

$u = \sqrt{x}+5 \qquad du = \frac{1}{2\sqrt{x}}dx$

$\hookrightarrow \sqrt{x} = u-5$

$$= \int \frac{2\sqrt{x}}{\sqrt{x}+5}\frac{dx}{2\sqrt{x}}$$

$$= \left.\int \frac{2(u-5)}{u}\, du\right|_{u=\sqrt{x}+5} = \left.2\int 1 - \frac{5}{u}\, du\right|_{u=\sqrt{x}+5}$$

$$= \left.2\left(u - 5\ln|u|\right)+C\right|_{u=\sqrt{x}+5} = \boxed{2\left((\sqrt{x}+5) - 5\ln|\sqrt{x}+5|\right)+C}$$

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Quiz number 3

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Find the following antiderivatives:

5. $\int x \sin x \, dx$

$u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$= -x\cos x - \int -\cos x \, dx = -x\cos x + \int \cos x \, dx$

$\boxed{= -x\cos x + \sin x + C}$

6. $\int \ln(1+x^2) \, dx$

$u = \ln(1+x^2) \quad dv = dx$

$du = \frac{2x}{1+x^2} dx \quad v = x$

$$\begin{array}{r} 2 \\ 1+x^2 \overline{)\, 2x^2 } \\ -\;\; 2x^2+2 \\ \hline -2 \end{array}$$

$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx$

$= x\ln(1+x^2) - \int 2 - \frac{2}{1+x^2} dx$

$\boxed{= x\ln(1+x^2) - (2x - 2\operatorname{Arctan} x) + C}$

$= x\ln(1+x^2) - 2x + 2\operatorname{Arctan} x + C$

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Quiz number 36

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Find the following antiderivatives:

5. $\int x \sec^2 x \, dx$ $=$

$u = x \quad dv = \sec^2 x \, dx$

$du = dx \quad v = \tan x$

$u = \cos x$

$du = -\sin x \, dx$

$= x\tan x - \int \tan x \, dx = x\tan x - \int \frac{\sin x}{\cos x} dx$

$= x\tan x + \int \frac{-\sin x \, dx}{\cos x} = x\tan x + \int \frac{du}{u}\Big|_{u=\cos x}$

$= x\tan x + \left(\ln|u| + C\Big|_{u=\cos x}\right) \boxed{= x\tan x + \ln|\cos x| + C}$

6. $\int xe^x \, dx$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$= xe^x - \int e^x dx = \boxed{xe^x - e^x + C}$

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Quiz number 4

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

7. Find the following integral:

$$\int \sec^5 x \tan^3 x \, dx$$

$du = \sec x \tan x \, dx$

$u = \sec x$

$= \int \sec^4 x \tan^2 x (\sec x \tan x \, dx)$

$= \int \sec^4 x (\sec^2 x - 1)(\sec x \tan x \, dx)$

$= \int u^4(u^2-1)\,du \Big|_{u=\sec x} = \int u^6 - u^4 \, du \Big|_{u=\sec x}$

$= \frac{u^7}{7} - \frac{u^5}{5} + C \Big|_{u=\sec x} = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C.$

8. Using a trigonometric substitution, convert the following integral into a trigonometric integral. **You do not need to solve the resulting integral** (unless you are bored...).

$$(*) = \int \frac{dx}{x^2\sqrt{x^2+4}}.$$

$\tan^2 x + 1 = \sec^2 x$

$4\tan^2 x + 4 = 4\sec^2 x$

$(2\tan x)^2 + 4 = (2\sec x)^2$

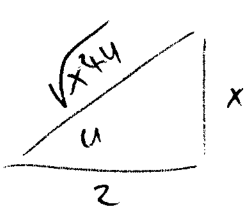
$x = 2\tan u$

$dx = 2\sec^2 u \, du$

$(*) = \int \frac{2\sec^2 u \, du}{(2\tan u)^2(2\sec u)} \Big|_{x=2\tan u} = \boxed{\frac{1}{4}\int \frac{\sec u}{\tan^2 u}\,du \Big|_{x=2\tan u}}$

$= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\,du \Big|_{x=2\tan u} = \frac{1}{4}\int \csc u \cot u \, du \Big|_{x=2\tan u}$

$= -\frac{1}{4}\csc u + C \Big|_{x=2\tan u} = \boxed{-\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + C}$!



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Math 107H, Section 3

Quiz number 5

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

9. Find the (implicit) solutions to the differential equation

$$x\frac{dy}{dx} = y^2 - 1$$

writing this as $\frac{dy}{y^2-1} = \frac{dx}{x}$ and integrating, we have

$\int \frac{dx}{x} = \ln|x| + C$, and

$$\int \frac{dy}{y^2-1} = \int \frac{dy}{(y-1)(y+1)} = \frac{1}{2}\int \frac{1}{y-1} + \frac{1}{y+1}\,dy = \frac{1}{2}\left(\ln|y-1| + \ln|y+1|\right) + C$$

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} = \frac{A(y+1)+B(y-1)}{(y-1)(y+1)}$$

$A(y+1) + B(y-1) = 1$ $\quad y=1:\ 2A=1 \quad A=\frac{1}{2}$

$y=-1:\ -2B=1 \quad B=-\frac{1}{2}$

So $\boxed{\ln|x| = \frac{1}{2}\left(\ln|y-1| + \ln|y+1|\right) + C}$

$$e^{\ln|x|} = |x| = |y-1|^{1/2}|y+1|^{1/2}\cdot e^{C}$$

$$|x| = k|y^2-1|^{1/2}$$

$$x^2 = k^2(y^2-1)$$

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Math 107H, Section 3

Quiz number 6

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

10. *Use the integral test* to decide if the following series converges:

$$\sum_{n=0}^{\infty} \frac{n}{e^n}$$

[Don't forget to check that we *can* use the integral test!]

$a_n = \frac{n}{e^n} = ne^{-n} = f(n)$ for $f(x) = xe^{-x}$

$n \geq 0$ and $e^{-n} \geq 0$ so $a_n \geq 0$.

$f'(x) = e^{-x} + x(-e^{-x})$
$= e^{-x} - xe^{-x} = (1-x)e^{-x}$
< 0 for $x > 1$

So f is eventually decreasing.

$u = x \quad dv = e^{-x}dx$
$du = dx \quad v = -e^{-x}$

Then $\int_1^\infty xe^{-x}\,dx = -xe^{-x}\Big|_1^\infty - \int_1^\infty -e^{-x}\,dx = -xe^{-x}\Big|_1^\infty + \int_1^\infty e^{-x}\,dx = -xe^{-x}\Big|_1^\infty - e^{-x}\Big|_1^\infty$

$= \lim_{N\to\infty} (-xe^{-x} - e^{-x})\Big|_1^N = \lim_{N\to\infty} (-Ne^{-N} - e^{-N}) - (-1e^{-1} - e^{-1})$

$= 2e^{-1} - \lim_{N\to\infty} Ne^{-N} + e^{-N}$

BA! $Ne^{-N} = \frac{N}{e^N}$ and $N \to \infty$, $e^N \to \infty$ as $N \to \infty$, so use

L'Hopital: $\lim_{N\to\infty} \frac{N}{e^N} = \lim_{N\to\infty} \frac{(1)}{(e^N)} = 0$ since $e^N \to \infty$ as $N \to \infty$

So $\lim_{N\to\infty} Ne^{-N} + e^{-N} = 0 + 0 = \underline{\underline{0}}$. So $\int_1^\infty xe^{-x}\,dx = 2e^{-1} - 0 = 2e^{-1} < \infty$

So $\sum ne^{-n}$ converges by the integral test.

Note: We can also integrate xe^{-x} by u-subs:

$\int xe^{-x}\,dx \overset{u\text{-}}{=} \int \frac{x}{e^x}\,dx \quad u = e^x \quad du = e^x dx \quad x = \ln u$

$v = \ln u \quad dw = \frac{du}{u^2}$

$dv = \frac{1}{u}du \quad w = \frac{-1}{u}$

$= \int \frac{x}{(e^x)^2}(e^x dx) = \int \frac{\ln u}{u^2}\,du \Big|_{u=e^x} = \frac{-\ln u}{u}\Big|_{u=e^x} + \int \frac{du}{u^2}\Big|_{u=e^x} = \ldots$

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Math 107H Section 3

Quiz number 7

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

11. Determine the convergence or divergence of the following series:

(a) $\sum_{n=0}^{\infty} \frac{n^{\frac{4}{3}}-1}{\sqrt{n^3+7}}$ 'looks like' $\frac{n^{4/3}}{\sqrt{n^3}} = \frac{n^{4/3}}{n^{3/2}} = n^{4/3-3/2} = n^{\frac{8-9}{6}} = n^{-1/6} = \frac{1}{n^{1/6}}$

$\boxed{\text{limit compare:}}$ $\frac{n^{4/3}-1}{\sqrt{n^3+7}} \Big/ \frac{n^{4/3}}{\sqrt{n^3}} = \frac{n^{4/3}-1}{n^{4/3}} \cdot \sqrt{\frac{n^3}{n^3+7}} = \frac{1-n^{-4/3}}{1} \cdot \sqrt{\frac{1}{1+7/n^3}}$

$\to \frac{1-0}{1} \cdot \sqrt{\frac{1}{1+0}} = 1 \cdot 1 = 1$ as $n \to \infty$ $1 \neq 0$ so since

$\sum \frac{1}{n^{1/6}}$ diverges (p-series, $p = 1/6 < 1$), $\boxed{\sum_{n=0}^{\infty} \frac{n^{4/3}-1}{\sqrt{n^3+7}} \text{ diverges}}$.

(b) $\sum_{n=1}^{\infty} \frac{n^3 11^n}{n!} = \sum a_n$ $n!$ looks like a good candidate for $\boxed{\text{ratio test.}}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3 11^{n+1}}{(n+1)!} \Big/ \frac{n^3 11^n}{n!} = \frac{(n+1)^3 11^{n+1}}{(n+1)!} \frac{n!}{n^3 11^n}$

$= \frac{(n+1)^3}{n^3} \cdot \frac{11^{n+1}}{11^n} \cdot \frac{n!}{(n+1)!} = \left(\frac{n+1}{n}\right)^3 \cdot 11 \cdot \frac{1}{(n+1)}$

$= \left(\frac{1+1/n}{1}\right)^3 \cdot 11 \cdot \left(\frac{1}{n+1}\right) \to \left(\frac{1+0}{1}\right)^3 \cdot 11 \cdot 0 = 11 \cdot 0 = 0$

as $n \to \infty$ $0 < 1$ so $\boxed{\sum_{n=1}^{\infty} a_n \text{ converges}}$

Quiz #8 Solutions

12(a): $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n(4n+1)}(x+5)^n = \sum_{n=0}^{\infty} a_n \qquad a_n = \frac{n!}{3^n(4n+1)}(x+5)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\,|x+5|^{n+1}}{3^{n+1}(4(n+1)+1)} \cdot \frac{3^n(4n+1)}{n!\,|x+5|^n} = \frac{(n+1)!}{n!}\,\frac{3^n}{3^{n+1}}\,\frac{4n+1}{4n+5}\,|x+5|$$

$$= (n+1)\frac{1}{3}\underbrace{\frac{4n+1}{4n+5}}_{\to 1}|x+5| \xrightarrow{n\to\infty} \infty \quad (\text{since } (n+1)\to\infty)$$

unless $|x+5|=0$, so

$\displaystyle\sum_{n=0}^{\infty} a_n$ converges only if $|x+5|=0$,

so radius of convergence $= \underline{0}$.

12(b): $\displaystyle\sum_{n=0}^{\infty} (-1)^n n^4 (x-2)^n = \sum_{n=0}^{\infty} a_n \qquad a_n = (-1)^n n^4 (x-2)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^4(x-2)^{n+1}}{(-1)^n\, n^4\,(x-2)^n}\right| = \left|\frac{(-1)}{1}\right|\left(\frac{n+1}{n}\right)^4|x-2|$$

$$= \Bigg(\underbrace{\frac{n+1}{n}}_{\to 1}\Bigg)^4|x-2| \longrightarrow 1^4\cdot|x-2| = |x-2| < 1$$

$$\iff |x-2|<1 \ (!)$$

so radius of convergence is $\underline{1}$.

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Math 107H, Section 3

Quiz number 9

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

13. Express the (Cartesian) line $y = x + 5$ as a polar curve $r = f(\theta)$ for the appropriate function f.

    What is the range of angles θ that we use to describe the line (i.e., which directions do ~~to~~ we look, in order to "see" the line?)?

    [Hint: what value(s) does θ tend to when $r \to \infty$?]

$y = r\sin\theta$, $x = r\cos\theta$, so $y = x+5$ becomes

$r\sin\theta = r\cos\theta + 5$, so $r\sin\theta - r\cos\theta = r(\sin\theta - \cos\theta) = 5$

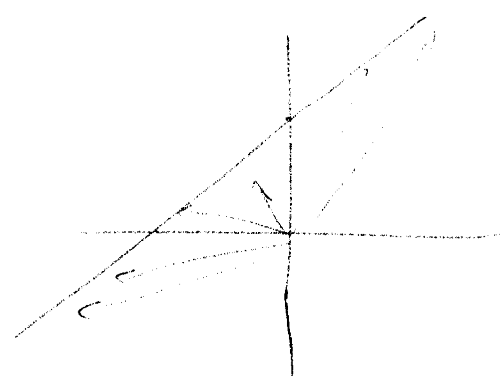
so $\boxed{r = \dfrac{5}{\sin\theta - \cos\theta}}$

For $r \to \infty$, since 5 stays constant, we would need $\sin\theta - \cos\theta \to 0$, so $\sin\theta = \cos\theta$, so $\dfrac{\sin\theta}{\cos\theta} = \tan\theta = 1$

so $\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \ldots$

So to see the line, we look in the directions between

$\boxed{\dfrac{\pi}{4} \text{ and } \dfrac{5\pi}{4}}$



Or: from $-\dfrac{3\pi}{4}$ to $\dfrac{\pi}{4}$ you are looking away but walking backwards!

Basically, other than the directions $\dfrac{\pi}{4} + k\pi$, $k = 0, \pm1, \pm2, \ldots$ you are looking "at" the line...