Math 107H, Section 3

Practice Exam 3 solutions

1. Show that the alternating series $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n)}$ converges, and determine a value of N so that the partial sum $\displaystyle\sum_{n=2}^{N}\frac{(-1)^n}{\ln(n)}$ is within .001 of the infinite sum.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n)} = \sum_{n=2}^{\infty}(-1)^n a_n \text{ for } a_n = \frac{1}{\ln(n)} = f(n) \text{ for } f(x) = \frac{1}{\ln(x)}.$$

Since $a_n > 0$ and $f'(x) = \dfrac{-1}{x(\ln x)^2} < 0$ for $x > 1$, a_n is decreasing, and $a_n \to 0$ as $n \to \infty$, since $\ln n \to \infty$ as $n \to infty$. So the series satisfies all of the conditions of the alternating series test, so the series converges.

In particular, the N-th partial sum, $\displaystyle\sum_{n=2}^{N}\frac{(-1)^n}{\ln(n)}$, is within $a_{N+1} = \dfrac{1}{\ln(N+1)}$ of the infinite sum. So to be within .001, we would like to choose N so that $\dfrac{1}{\ln(N+1)} < .001$, so $\ln(N+1) > (1/.001) = 1000$, so $N+1 > e^{1000}$. So we can choose N to be any number larger than the whole number part of e^{1000} (which is a pretty huge number, really....)

2. Compute the radius of convergence of the following power series:

$$f(x) = \sum_{n=0}^{\infty}\frac{2^n-1}{(n+4)^2}(x-3)^n = \sum a_n(x-3)^n$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}-1}{((n+1)+4)^2}}{\frac{2^n-1}{(n+4)^2}} = \frac{2^{n+1}-1}{2^n-1}\frac{(n+4)^2}{(n+5)^2} = \frac{2-2^{-n}}{1-2^{-n}}\frac{(1+\frac{4}{n})^2}{(1+\frac{5}{n})^2}, \text{ and since}$$

$\dfrac{1}{n} \to 0$ and $2^{-n} \to 0$ as $n \to \infty$,

$\dfrac{a_{n+1}}{a_n} = \dfrac{2-2^{-n}}{1-2^{-n}}\dfrac{(1+\frac{4}{n})^2}{(1+\frac{5}{n})^2} \to \dfrac{2-0}{1-0}\dfrac{(1+4\cdot 0)^2}{(1+5\cdot 0)^2} = 2\cdot 1 = 2 = L$, so the radius of convergence of $\sum a_n(x-3)^n$ is $R = \dfrac{1}{L} = \dfrac{1}{2}$.

3. Using the Taylor series $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty}x^n$, find a power series representation for the function

$$f(x) = \frac{x^2}{1+x^4}$$

centered at $x = 0$ (by an appropriate substitution and multiplication). Use this to find a series which converges to the integral

$$\int_0^{1/3} \frac{x^2}{1+x^4}\,dx\ .$$

$$f(x) = \frac{x^2}{1+x^4} = x^2\frac{1}{1+x^4} = x^2\frac{1}{1-(-x^4)}$$
$$= x^2\sum_{n=0}^{\infty}(-x^4)^n = x^2\sum_{n=0}^{\infty}(-1)^n x^{4n} = \sum_{n=0}^{\infty}(-1)^n x^{4n+2},$$

so $\displaystyle\int \frac{x^2}{1+x^4}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{4n+2}\,dx = c + \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}x^{4n+3}.$

So by the Fundamental Theorem of Calculus,

$$\int_0^{1/3} \frac{x^2}{1+x^4}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}x^{4n+3}\Big|_0^{1/3}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}\Big(\frac{1}{3}\Big)^{4n+3} - \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}0^{4n+3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}\Big(\frac{1}{3}\Big)^{4n+3}$$

[FYI: a somewhat laborious partial fractions decomposition will demonstrate that

$$\int \frac{x^2}{1+x^4}\,dx$$
$$= \frac{-\sqrt{2}}{8}\ln\Big(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\Big) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x+1) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x-1) + C]$$

4. Find the Taylor polynomial of degree 3, centered at $x = 8$, for the function

$$f(x) = x^{2/3}$$

and estimate the error in using your polynomial to approximate $f(7) = 7^{2/3}$.

To find the Taylor polynomial, we need derivatives:

$f(x) = x^{2/3}$
$f'(x) = (2/3)x^{-1/3}$
$f''(x) = (-1/3)(2/3)x^{-4/3}$
$f'''(x) = (-4/3)(-1/3)(2/3)x^{-7/3}$

Evaluating at $x = 8$, we get

$f(8) = 8^{2/3} = 2^2 = 4$
$f'(8) = (2/3)8^{-1/3} = (2/3)(1/2) = 1/3$
$f''(8) = (-1/3)(2/3)8^{-4/3} = (-1/3)(2/3)2^{-4} = (-2/9)(1/16) = -1/72$
$f'''(8) = (-4/3)(-1/3)(2/3)8^{-7/3} = (8/27)2^{-7} = (1/27)2^{-4} = 1/(27\cdot 16)$

So the degree 3 Taylor polynomial is

$$P_3(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 + \frac{f'''(8)}{3!}(x-8)^3$$

$$= 4 + \frac{1}{3}(x-8) + \frac{-1}{72\cdot 2}(x-8)^2 + \frac{1}{27\cdot 16\cdot 6}(x-8)^3$$

For the error term, we need the fourth derivative:

$f''''(x) = (-7/3)(-4/3)(-1/3)(2/3)x^{-10/3}$

We know that the remainder $R_3(x) = f(x) - P_3(x)$ satisfies $|R_3(7)| \le M\frac{|7-8|^4}{4!}$ where M is the largest value of $|f''''(x)|$ for x between 8 and 7. But $x^{-10/3}$ is a decreasing function, so its largest value will occur at the left endpoint, 7, so

$|R_3(7)| \le (-7/3)(-4/3)(-1/3)(2/3)7^{-10/3}\frac{|7-8|^4}{4!}$ (whatever that is...).

5. Express the polar equation $r = \sin(3\theta)$ as an equation in Cartesian coordinates.
[Hint: $\sin(3\theta) = \sin(\theta = 2\theta)$...]

$\sin(3\theta) = \sin(\theta = 2\theta) = \sin(\theta)\cos(2\theta) + \cos(\theta)\sin(2\theta) = \sin(\theta)(\cos^2(\theta) - \sin^2(\theta)) + \cos(\theta)(2\sin(\theta)\cos(\theta)) = 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)$

All of these trig functions would like an 'r' (so $x = r\cos(\theta)$ and $y = r\sin(\theta)$), and so

$r = \sin(3\theta)$ implies $r^4 = r^3(3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)) = 3(r\sin(\theta))(r\cos(\theta))^2 - (r\sin(\theta))^3$, so

$(x^2+y^2)^2 = (r^2)^2 = r^4 = 3(r\sin(\theta))(r\cos(\theta))^2 - (r\sin(\theta))^3 = 3yx^2 - y^3$,

so an equation in Cartesian coordinates is given by $(x^2+y^2)^2 = 3yx^2 - y^3$.

[Note that $3\sin(\theta)\cos^2(\theta) - \sin^3(\theta) = 3\sin(\theta)(1-\sin^2(\theta)) - \sin^3(\theta) = 3\sin(\theta) - 4\sin^3(\theta)$, so the ultimate answer can be written slightly differently, as $(x^2+y^2)^2 = 3y(x^2+y^2) - 4y^3$.]

6. Find the (rectangular) equation of the line tangent to the graph of the polar curve

$$r = 3\sin\theta - \cos(3\theta)$$

at the point where $\theta = \frac{\pi}{4}$.

$r = 3\sin\theta - \cos(3\theta) = f(\theta)$, so
$x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so

$$dy/dx = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} .$$

But $f'(\theta) = 3\cos\theta + 3\sin(3\theta)$, so
$f'(\pi/4) = 3\cos(\pi/4) + 3\sin(3\pi/4) = 3(\sqrt{2}/2) + 3(-\sqrt{2}/2) = 0$, and
$f(\pi/4) = 3\sin(\pi/4) - \cos(3\pi/4) = 3(\sqrt{2}/2) - (-\sqrt{2}/2) = 4(\sqrt{2}/2) = 2\sqrt{2}$.
and since $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$, we have, at $\theta = \pi/4$,

$$dy/dx = \frac{f'(\pi/4)\sin(\pi/4) + f(\pi/4)\cos(\pi/4)}{f'(\pi/4)\cos(\pi/4) - f(\pi/4)\sin(\pi/4)} = \frac{(0)(\sqrt{2}/2) + (2\sqrt{2})(\sqrt{2}/2)}{(0)(\sqrt{2}/2) - (2\sqrt{2})(\sqrt{2}/2)}$$

$$= \frac{(2\sqrt{2})(\sqrt{2}/2)}{-(2\sqrt{2})(\sqrt{2}/2)} = -1.$$

So the slope of the tangent line is -1, and it goes through the point $(x,y) = (f(\pi/4)\cos(\pi/4), f(\pi/4)\sin(\pi/4)) = ((2\sqrt{2})(\sqrt{2}/2),(2\sqrt{2})(\sqrt{2}/2)) = (2,2)$, so the equation for the tangent line is $y-2 = (-1)(x-2)$, or $y = -x+4$.

7. Find the length of the polar curve $r = \theta^2$ from $\theta = 0$ to $\theta = 2\pi$.

For $r = \theta^2 = f(\theta)$, since
$(dx/d\theta)^2 + (dy/d\theta)^2 = (f(\theta))^2 + (f'(\theta))^2 = (\theta^2)^2 + (2\theta)^2 = \theta^4 + 4\theta^2 = \theta^2(\theta^2+4)$,
we have

$$\text{Length} = \int_0^{2\pi} \sqrt{\theta^2(\theta^2+4)}\ d\theta = \int_0^{2\pi} \sqrt{\theta^2}\sqrt{\theta^2+4}\ d\theta$$
$$= \int_0^{2\pi} |\theta|\sqrt{\theta^2+4}\ d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\ d\theta$$

Setting $u = \theta^2+4$, then $du = 2\theta\ d\theta$, and for
$\theta = 0$, $u = 4$, while for $\theta = 2\pi$, $u = 4\pi^2+4$, so

$$\text{Length} = \frac{1}{2}\int_4^{4\pi^2+4} \sqrt{u}\ du = (\frac{1}{2})(\frac{2}{3})u^{3/2}\Big|_4^{4\pi^2+4}$$
$$= \frac{1}{3}((4\pi^2+4)^{3/2} - 4^{3/2}) = \frac{8}{3}((\pi^2+1)^{3/2} - 1)$$

8. Find the area inside of the graph of the polar curve

$$r = \sin(\theta) - \cos(\theta)$$

from $\theta = \dfrac{\pi}{4}$ to $\theta = \dfrac{5\pi}{4}$.

[Extra credit: What does this curve look like? (Hint: multiply both sides by r.)]

Since Area$=\displaystyle\int \frac{1}{2}(f(\theta))^2\ d\theta$, we have

$$\text{Area} = \frac{1}{2}\int_{\pi/4}^{5\pi/4} (\sin\theta - \cos\theta)^2\ d\theta = \frac{1}{2}\int_{\pi/4}^{5\pi/4} \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta\ d\theta$$
$$= \frac{1}{2}\int_{\pi/4}^{5\pi/4} 1 - 2\sin\theta\cos\theta\ d\theta = \frac{1}{2}\int_{\pi/4}^{5\pi/4} 1 - \sin(2\theta)\ d\theta = \frac{1}{2}[\theta + \frac{1}{2}\cos(2\theta)]\Big|_{\pi/4}^{5\pi/4}$$
$$= \frac{1}{2}[(5\pi/4 + \frac{1}{2}\cos(5\pi/2)) - (\pi/4 + \frac{1}{2}\cos(\pi/2))] = \frac{1}{2}[(5\pi/4 + \frac{1}{2}) - (\pi/4 + \frac{1}{2})]$$
$$= \frac{1}{2}[(5\pi/4) - (\pi/4)] = \frac{1}{2}[\pi] = \frac{\pi}{2}$$

To see what this curve is, we have $r = \sin(\theta) - \cos(\theta)$, so $r^2 = r\sin(\theta) - r\cos(\theta)$, so $x^2+y^2 = y-x$, so $(x^2+x)+(y^2-y) = 0$, so $(x^2+x+\dfrac{1}{4})+(y^2-y+\dfrac{1}{4}) = \dfrac{1}{4}+\dfrac{1}{4} = \dfrac{1}{2}$,
so $(x+\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{2} = (\frac{1}{\sqrt{2}})^2$

This is a circle, centered at $(-\dfrac{1}{2},\dfrac{1}{2})$, with radius $\dfrac{1}{\sqrt{2}}$!