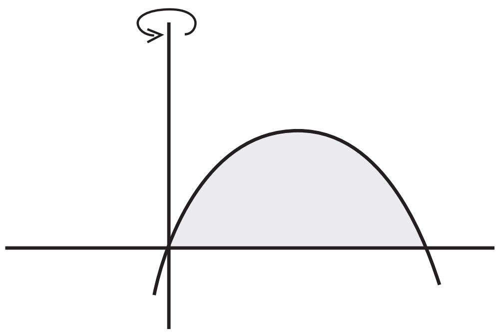
Math 107H Practice Exam 2 Solutions

Note: Most sequences/series can be shown to converge or diverge in more than one way; the solutions below illustrate only one such method. Your approach may differ....

1. Find the volume of the region obtained by revolving the region under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi$ around the y-axis (see figure).



By cylindrical shells: radius $= x$, height $= \sin x$, so

Volume $= \displaystyle\int_0^\pi 2\pi x \sin x \; dx = 2\pi \int_0^\pi x \sin x \; dx$.

$\displaystyle\int_0^\pi x \sin x \; dx = (****)$; integrating by parts,

$u = x$, $dv = \sin x \; dx$, so $du = dx$ and $v = -\cos x$, so

$(****) = -x\cos x - \displaystyle\int -\cos x \; dx = -x\cos x + \int \cos x \; dx = -x\cos x + \sin x + c$

So Volume $= 2\pi(-x\cos x + \sin x)\Big|_0^\pi = 2\pi[(-\pi(-1)+0) - (0(1)-0)] = 2\pi^2$.

noindent **2.** Find the improper integral $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3} \; dx$.

$\displaystyle\int \frac{1}{x(\ln x)^3} \; dx = \int \frac{du}{u^3}\Big|_{u=\ln x}$ via the u-substitution $u = \ln x$, so $du = \dfrac{1}{x} \; dx$,

which equals $\displaystyle\int u^{-3} \; du\Big|_{u=\ln x} = -\frac{1}{2}u^{-2} + c\Big|_{u=\ln x} = -\frac{1}{2(\ln x)^2} + c$

So $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3} \; dx = \lim_{n\to\infty} \int_2^N \frac{1}{x(\ln x)^3} \; dx$

$\displaystyle = \lim_{n\to\infty} -\frac{1}{2(\ln x)^2}\Big|_2^N = \lim_{n\to\infty} \frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln N)^2}$

But since $\ln N \to \infty$ as $N \to \infty$, $\dfrac{1}{2(\ln N)^2} \to 0$ as $N \to \infty$, so

3. Determine the convergence or divergence of the following sequences:

(a) $a_n = \dfrac{n^3 + 6n^2 \ln n - 1}{2 - 3n^3} = \dfrac{1 + 6(\ln n)/n - 1/n^3}{2/n^3 - 3}$.
and since $1/n^3 \to 0$ and $(\ln n)/n \to 0$ as $n \to \infty$,

$$a_n \to \frac{1 + 6 \cdot 0 - 0}{2 \cdot 0 - 3} = \frac{1}{-3} = -\frac{1}{3} \text{ as } n \to \infty.$$

(b) $b_n = \dfrac{n^{n+\frac{1}{n}}}{(n+3)^n}$

$$b_n = \frac{n^{n+\frac{1}{n}}}{(n+3)^n} = \frac{n^n n^{\frac{1}{n}}}{(n+3)^n} = \frac{n^{\frac{1}{n}}}{\left(\frac{n+3}{n}\right)^n} = \frac{n^{\frac{1}{n}}}{\left(1 + \frac{3}{n}\right)^n} .$$

But $n^{\frac{1}{n}} \to 1$ and $\left(1 + \dfrac{3}{n}\right)^n \to e^3$ as $n \to \infty$, so $b_n \to \dfrac{1}{e^3} = e^{-3}$ as $n \to \infty$.

4. Determine the convergence or divergence of the following series:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)(\ln n)^{2/3}}$ [Hint: limit compare, then integral...]

$a_n = \dfrac{1}{(n-1)(\ln n)^{2/3}}$ looks like $b_n = \dfrac{1}{n(\ln n)^{2/3}}$, and $\dfrac{a_n}{b_n} = \dfrac{n}{n-1} \to 1$ as $n \to \infty$,
so $\sum a_n$ converges precisely when $\sum b_n$ converges. But:
$b_n = \dfrac{1}{n(\ln n)^{2/3}} = f(n)$ for $f(x) = \dfrac{1}{x(\ln x)^{2/3}}$, which is continuous and decreasing (x and $\ln(x)$ are both increasing, so $(\ln x)^{2/3}$ is increasing, so their reciprocals are decreasing, and so the product is decreasing). So we can apply the integral test:

$\displaystyle\int \frac{1}{x(\ln x)^{2/3}}\, dx = \int \frac{du}{u^{2/3}}\, du|_{u=\ln x} = 3u^{1/3}|_{u=\ln x} = 3(\ln x)^{1/3}$, so

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\, dx = \lim_{N\to\infty} [3(\ln N)^{1/3} - 3(\ln 2)^{1/3}]$, but since $\ln N \to \infty$ as $N \to \infty$,

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\, dx$ diverges, so $\sum b_n$ diverges, so $\sum a_n$ **diverges**.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ $\quad a_n = \dfrac{6n}{(1-n^2)^2}$ looks like $b_n = \dfrac{6n}{(-n^2)^2} = \dfrac{6}{n^3}$, which converges:

So note that $\dfrac{a_n}{b_n} = \dfrac{(-n^2)^2}{(1-n^2)^2} = \dfrac{1}{(\frac{1}{n^2} - 1)^2}$, and since $1/n^2 \to 0$ as $n \to \infty$, $\dfrac{a_n}{b_n} \to \dfrac{1}{(0-1)^2} = 1$ as $n \to \infty$, so by limit comparison, $\sum a_n$ converges precisely when $\sum b_n$ converges.

But: $\sum b_n = \sum \frac{6}{n^3} = 6\sum \frac{1}{n^3}$, which converges ($p$-series, $p = 3 > 1$), so $\sum b_n$ converges, so $\sum a_n = \sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ **converges**.

5. Determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{(n-1)!}{2^n n^3}$ $\qquad \sum_{n=1}^{\infty} \frac{(n-1)!}{2^n n^3} = \sum a_n$ and

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((n+1)-1)!}{2^{n+1}(n+1)^3}}{\frac{(n-1)!}{2^n n^3}} = \frac{n!}{(n-1)!}\frac{2^n}{2^{n+1}}\frac{n^3}{(n+1)^3} = (n)\left(\frac{1}{2}\right)\left(\frac{n}{n+1}\right)^3.$$

Since $\frac{n}{n+1} \to 1$ and $n \to \infty$ as $n \to \infty$, $\frac{a_{n+1}}{a_n} = (\text{big})(\frac{1}{2})(\text{close to } 1)$, which is big, as n gets large, so $\frac{a_{n+1}}{a_n} \to \infty$ as $n \to \infty$, so $\sum a_n$ **diverges** by the Ratio Test.

(b) $\sum_{n=0}^{\infty} \frac{n2^{2n+1}}{9^n+1}$ $\qquad \sum_{n=0}^{\infty} \frac{n2^{2n+1}}{9^n+1} = \sum a_n$ and

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)2^{2(n+1)+1}}{9^{n+1}+1)}}{\frac{n2^{2n+1}}{9^n+1}} = \frac{(n+1)}{n}\frac{2^{2(n+1)+1}}{2^{2n+1}}\frac{9^n+1}{9^{n+1}+1} = \frac{(1+1/n)}{1}\frac{2^{2n+3}}{2^{2n+1}}\frac{1+9^{-n}}{9+9^{-n}} =$$

$(1+\frac{1}{n})(2^2)\frac{1+9^{-n}}{9+9^{-n}}$, and since $\frac{1}{n} \to 0$ and $9^{-n} \to 0$ as $n \to \infty$,

$$\frac{a_{n+1}}{a_n} = (1+\frac{1}{n})(2^2)\frac{1+9^{-n}}{9+9^{-n}} \to (1+0)(4)\frac{1+0}{9+0} = \frac{4}{9} < 1,$$

so $\sum_{n=0}^{\infty} \frac{n2^{2n+1}}{9^n+1} = \sum a_n$ **converges** by the ratio test.

6. Set up, **but do not evaluate**, the integral which will compute the arclength of the graph of $y = x\sqrt{1+x^2}$ from $x = 0$ to $x = 3$.

$f(x) = x\sqrt{1+x^2} = x(1+x^2)^{\frac{1}{2}}$, so $f'(x) = (1+x^2)^{\frac{1}{2}} + x(\frac{1}{2})(1+x^2)^{-\frac{1}{2}}(2x) = (1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}$.

So Arclength $= \int_0^3 \sqrt{1+[f'(x)]^2}\,dx = \int_0^3 \sqrt{1+[(1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}]^2}\,dx$

Solutions

1. (20 pts.) Cesium-137, denoted Cs_{137}, is a radioactive substance with a half-life of 30 years. That is, if $C(t)$ represents the amount of Cs_{137} in a sample after t years, then

$$C(30) = \frac{1}{2}C(0) .$$

If we start with a 4 gram sample of Cs_{137}, how much Cs_{137} will remain after 10 years?

$$C(t) = C(0)e^{kt} = 4e^{kt}$$

$$C(30) = \frac{1}{2}C(0) = 2 = 4e^{30k}$$

$$\Rightarrow \frac{1}{2} = e^{30k} \quad \ln(\tfrac{1}{2}) = 30k \quad k = \frac{1}{30}\ln(\tfrac{1}{2})$$

$$\text{so} \quad C(t) = 4e^{\ln(\frac{1}{2})\frac{t}{30}}, \quad \text{so}$$

$$C(10) = \boxed{4e^{\ln(\frac{1}{2})\frac{10}{30}}}$$

$$= 4e^{\ln((\frac{1}{2})^{1/3})} = 4(\tfrac{1}{2})^{1/3}$$

Solutions

Section 3

Exam 1

Exams provide you, the student, with an opportunity to demonstrate your understanding of the techniques presented in the course. So:

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

8. Find the (implicit) solution to the separable differential equation

$$\frac{dy}{dt} = \frac{te^{y+t}}{y}$$

$$\frac{dy}{dt} = \left(\frac{e^y}{y}\right)\left(te^t\right)$$

$$\int ye^{-y}dy = \int te^t\,dt$$

$u=y \quad dv=e^{-y}dy$

$du=dy \quad v=-e^{-y}$

$u=t \quad dv=e^t dt$

$du=dt \quad v=e^t$

$$-ye^{-y}+\int e^{-y}dy = te^t - \int e^t dt$$

$$-ye^{-y}-e^{-y} = te^t - e^t + C$$