Solutions

Name:

Math 107H Section 3

Final Exam

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Find the following integrals (10 pts each):

1-1: $\int \frac{x^3}{x-1}\,dx$

$$x-1 \overline{\big) x^3} = x^2+x+1+\frac{1}{x-1}$$
$$-(x^3-x^2),\quad x^2,\quad -(x^2-x),\quad x,\quad -(x-1),\quad 1$$

$$= \int x^2+x+1+\frac{1}{x-1}\,dx$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+C$$

$\boxed{\text{or}}$

$u = x-1$

$du = dx$

$x = u+1$

$$= \int \frac{(u+1)^3}{u}\,du \Big|_{u=x-1}$$

$$= \int \frac{u^3+3u^2+3u+1}{u}\,du \Big|_{u=x-1} = \int u^2+3u+3+\frac{1}{u}\,du \Big|_{u=x-1}$$

$$= \frac{u^3}{3}+\frac{3u^2}{2}+3u+\ln|u|+C \Big|_{u=x-1} = \frac{(x-1)^3}{3}+\frac{3(x-1)^2}{2}+3(x-1)+\ln|x-1|+C$$

1-2: $\int_1^3 \frac{x-1}{(x+1)(x+3)}\,dx$

$$= \int_1^3 \frac{A}{x+1}+\frac{B}{x+3}\,dx = \int_1^3 \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}\,dx$$

$A(x+3)+B(x+1) = x-1$

$x=-3$: $B(-2) = -4$, $B=2$

$x=-1$: $A(2) = -2$, $A=-1$

$$= \int_1^3 \frac{-1}{x+1}+\frac{2}{x+3}\,dx$$

$$= -\ln|x+1|+2\ln|x+3| \Big|_1^3$$

$$= (-\ln(4)+2\ln(6)) - (-\ln(2)+2\ln(4))$$

$$= -\ln 4+\ln(36)+\ln(2)-\ln(16) = \ln\left(\frac{36\cdot 2}{4\cdot 16}\right) = \ln\left(\frac{9\cdot 1}{2\cdot 4}\right) = \ln\left(\frac{9}{8}\right)$$

1-3: $\int x \arcsin x \; dx$

$$u = \arcsin x \qquad dv = x\,dx$$
$$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = \frac{1}{2}x^2$$

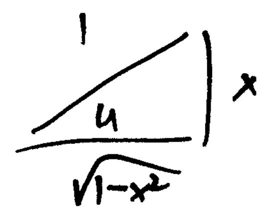
$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx \qquad \begin{array}{l} x = \sin u \\ dx = \cos u\,du \end{array} \qquad \sqrt{1-x^2} = \cos u$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{\sin^2 u}{\cos u}\cos u\,du \Big|_{x=\sin u}$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \sin^2 u\,du \Big|_{x=\sin u}$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\left(-\frac{1}{2}\sin u \cos u \Big|_{x=\sin u} + \frac{1}{2}\int \sin^0 u\,du\right)\Big|_{x=\sin u}$$

$$= \frac{1}{2}x^2 \arcsin x + \left(\frac{1}{4}\sin u \cos u - \frac{1}{4}\int du\right)\Big|_{x=\sin u}$$



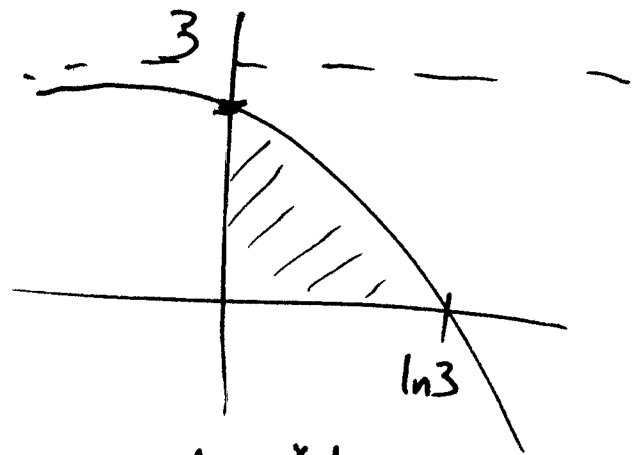
$$= \frac{1}{2}x^2 \arcsin x + \left(\frac{1}{4}\sin u \cos u - \frac{1}{4}u + C\right)\Big|_{x=\sin u}$$

$$\boxed{= \frac{1}{2}x^2 \arcsin x + \frac{1}{4}x\sqrt{1-x^2} - \frac{1}{4}\arcsin x + C}$$

2. (15 pts.) Find the volume of the solid obtained by revolving the region bewteen the x-axis, the y-axis, and the graph of the function

$$f(x) = 3 - e^x$$

around the y-axis.



$$\text{Volume} = 2\pi\int_0^{\ln 3} x(3-e^x)\,dx$$

$$= 2\pi\int_0^{\ln 3} 3x - xe^x\,dx$$

$u = x \quad dv = e^x\,dx$

$du = dx \quad v = e^x$

$$= 2\pi\left(\frac{3x^2}{2}\Big|_0^{\ln 3} - \int_0^{\ln 3} xe^x\,dx\right)$$

$$= 2\pi\left(\frac{3x^2}{2}\Big|_0^{\ln 3} - \left(xe^x\Big|_0^{\ln 3} - \int_0^{\ln 3} e^x\,dx\right)\right) = 2\pi\left(\frac{3x^2}{2}\Big|_0^{\ln 3} - xe^x\Big|_0^{\ln 3} + e^x\Big|_0^{\ln 3}\right)$$

$$= 2\pi\left(\left(\frac{3(\ln 3)^2}{2} - (\ln 3)e^{\ln 3} + e^{\ln 3}\right) - (0 - 0 + e^0)\right)$$

$$= 3\pi(\ln 3)^2 - 6\pi\ln 3 + 6\pi - 2\pi$$

$\boxed{\text{or}}$

$y = 3 - e^x \qquad e^x = 3 - y \qquad x = \ln(3-y)$

$x = 0 \to y = 3 - e^0 = 3 - 1 = 2$

$y = 2 \to u = 1$

$y = 0 \to u = 3$

$u = 3 - y$

$du = -dy$

$$\text{Volume} = \pi\int_0^{\cancel{3}\,2} (\ln(3-y))^2\,dy = \ldots$$

$$= -\pi\int_3^1 (\ln(u))^2\,du$$

$v = (\ln u)^2 \quad dw = du$

$dv = \frac{2\ln u}{u}\,du \quad w = u$

$$= \pi\left(u(\ln u)^2\Big|_3^1 - \int_3^1 2\ln u\,du\right) =$$

$v = \ln u \quad dw = 2\,du$

$dv = \frac{1}{u}\,du \quad w = 2u$

$$= -\pi\left(u(\ln u)^2\Big|_3^1 - \left(2u\ln u\Big|_3^1 - \int_3^1 2\,du\right)\right)$$

$$= -\pi\left(u(\ln u)^2 - 2u\ln u + 2u\right)\Big|_3^1$$

$$= -\pi\left((0 - 0 + 2) - (3(\ln 3)^2 - 6(\ln 3) + 6)\right)$$

$$= \boxed{3\pi(\ln 3)^2 - 6\pi\ln 3 + 6\pi - 2\pi}$$

3. Find the following improper integrals (10 pts. each):

3-1: $\displaystyle\int_1^\infty \frac{\ln x}{x}\, dx$ $= \lim_{N\to\infty} \int_1^N \frac{\ln x}{x} dx$ $u = \ln x$, $du = \frac{1}{x} dx$ $x=N$ $u = \ln N$, $x = 1$ $u = \ln 1 = 0$

$$= \lim_{N\to\infty} \int_0^{\ln N} u\,du = \lim_{N\to\infty} \left.\frac{u^2}{2}\right|_0^{\ln N} = \lim_{N\to\infty} \frac{(\ln N)^2}{2} - 0$$

$$= \lim_{N\to\infty} \frac{(\ln N)^2}{2} = \infty$$ diverges.

3-2: $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$ trouble at $x=1$.

$$= \lim_{a\to 1^-} \int_0^a \frac{1}{\sqrt{1-x}} dx$$ $u = 1-x$, $du = -dx$ $x=0$ $u=1$, $x=a$ $u=1-a$

$$= \lim_{a\to 1^-} -\int_1^{1-a} \frac{du}{\sqrt{u}} = \lim_{a\to 1^-} -\int_1^{1-a} u^{-\frac{1}{2}} du = \lim_{a\to 1^-} \left. -2u^{\frac{1}{2}} \right|_1^{1-a}$$

$$= \lim_{a\to 1^-} -2\sqrt{1-a} - (-2\sqrt{1}) = \lim_{a\to 1^-} 2 - 2\sqrt{1-a} = 2 - 2\sqrt{1-1}$$
$$= 2 - 2\sqrt{0} = \underline{\underline{2}}$$

(converges.)

4. Determine the convergence or divergence of each of the following series (10 pts. each):

4-1: $\sum_{n=0}^{\infty} \frac{e^n}{n!} = \sum_{n=0}^{\infty} a_n$ $\quad \frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \frac{e}{n+1} \to 0 < 1$

as $n \to \infty$

so converges by ratio(n) test.

4-2: $\sum_{n=2}^{\infty} \frac{n^2}{\sqrt{n^6+n-2}} = \sum a_n$ behaves like $\sum \frac{n^2}{\sqrt{n^6}} = \sum \frac{n^2}{n^3} = \sum \frac{1}{n}$

$b_n = 1/n$ $\quad \frac{a_n}{b_n} = \frac{\cancel{n^2}}{\sqrt{n^6+n-2}} \cdot \frac{\sqrt{n^6}}{\cancel{n^2}} = \sqrt{\frac{n^6}{n^6+n-2}}$

$= \sqrt{\frac{1}{1+n/n^6-2/n^6}} = \sqrt{\frac{1}{1+1/n^5-2/n^6}} \to \sqrt{\frac{1}{1+0+0}} = 1$ $\neq 0, \infty$

so since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 1 \leq 1$)

$\sum_{n=2}^{\infty} a_n$ also diverges by limit comparison.

5. (~~15~~ 20 pts.) For what values of x does the power series $\sum_{n=0}^{\infty} \frac{n\sqrt{n}}{2^n} x^{3n}$ converge?

$$= \sum_{n=0}^{\infty} a_n \qquad a_n = \frac{n\sqrt{n}}{2^n} x^{3n} = \frac{n^{3/2} x^{3n}}{2^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{3/2} x^{3(n+1)}}{2^{n+1}} \cdot \frac{2^n}{n^{3/2} x^{3n}}\right|$$

$$= \left|\frac{(n+1)^{3/2}}{n^{3/2}} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{x^{3n+3}}{x^{3n}}\right| = \left(\frac{n+1}{n}\right)^{3/2} \cdot \frac{1}{2} \cdot |x^3|$$

$$= \left(1+\frac{1}{n}\right)^{3/2} \cdot \frac{1}{2}|x|^3 \longrightarrow 1^{3/2} \cdot \frac{1}{2} \cdot |x|^3 = \frac{|x|^3}{2} < 1 \text{ for}$$

$|x|^3 < 2$, i.e. $|x| < 2^{1/3}$

So converges for $|x| < 2^{1/3}$, diverges for $|x| > 2^{1/3}$.

Check $|x| = 2^{1/3}$:

$x = 2^{1/3}$: $\sum a_n = \sum \frac{n^{3/2}(2^{1/3})^{3n}}{2^n} = \sum \frac{n^{3/2} 2^n}{2^n} = \sum n^{3/2}$

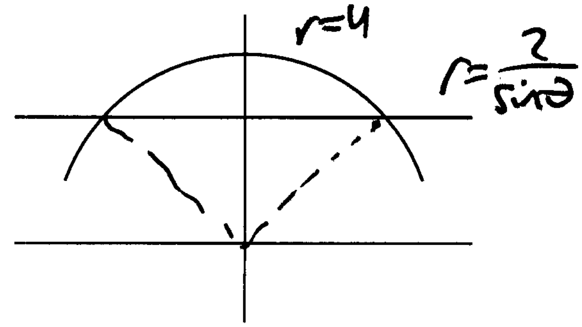
diverges, since $n^{3/2} \to \infty$, ~~not~~ $\not\to 0$ as $n \to \infty$.

$x = -2^{1/3}$: $\sum a_n = \sum \frac{n^{3/2}(-2^{1/3})^{3n}}{2^n} = \sum \frac{n^{3/2}(-1)^n 2^n}{2^n} = \sum (-1)^n n^{3/2}$

But $|(-1)^n n^{3/2}| = n^{3/2} \to \infty$ so $(-1)^n n^{3/2} \not\to 0$, so diverges.

So $\sum_{n=0}^{\infty} \frac{n\sqrt{n}\, x^{3n}}{2^n}$ converges for $|x| < 2^{1/3}$, diverges for all other x.

6. (15 pts.) Find the area lying between the polar curves $r = 4$ and $r = \dfrac{2}{\sin(\theta)}$ (see figure).



pts of intersection:

$4 = \frac{2}{\sin\theta} \qquad \sin\theta = \frac{2}{4} = \frac{1}{2}$

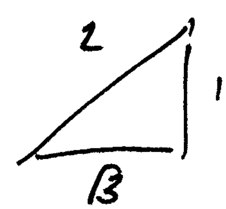
$\theta = \frac{\pi}{6}, \ \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\frac{2}{\sin\theta}$ is inside of 4 in this range, so:

$$\text{Area} = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4)^2 - \frac{1}{2}\left(\frac{2}{\sin\theta}\right)^2 d\theta = \int_{\pi/6}^{5\pi/6} 8 - \frac{2}{\sin^2\theta}\, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 8 - 2\csc^2\theta \, d\theta = 8\theta - 2(-\cot\theta)\Big|_{\pi/6}^{5\pi/6}$$

$$= 8\theta + 2\cot\theta\Big|_{\pi/6}^{5\pi/6} = \left(8\,\frac{5\pi}{6} + 2\cot\left(\frac{5\pi}{6}\right)\right) - \left(8\left(\frac{\pi}{6}\right) + 2\cot\left(\frac{\pi}{6}\right)\right)$$



$$= \left(\frac{20\pi}{3} + 2(-\sqrt{3})\right) - \left(\frac{4\pi}{3} + 2(\sqrt{3})\right)$$

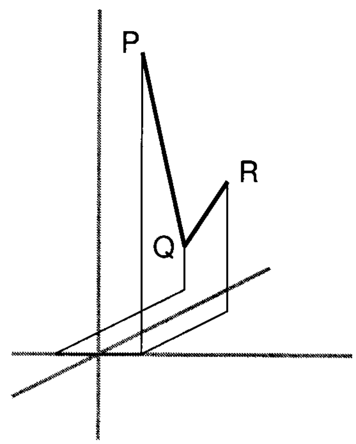
$$= \frac{16\pi}{3} - 4\sqrt{3}$$

15

7. (~~20~~ pts.) Find the orthogonal projection of the vector $\vec{v} = \vec{QP}$ onto the vector $\vec{w} = \vec{QR}$, for

$$P = (1, 0, 7),\ Q = (-1, 3, 1), \text{ and } R = (1, 2, 3).$$

~~Then find the shortest distance from the point P to the line through the points Q and R.~~



$$\vec{v} = \vec{QP} = (+2, -3, 6)$$

$$\vec{w} = \vec{QR} = (2, -1, 2)$$

$$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

$$= \frac{+4+3+12}{4+1+4}(2,-1,2)$$

$$= \frac{19}{9}(2,-1,2) = \left(\frac{\overset{38}{\cancel{22}}}{9}, \frac{-\overset{19}{\cancel{11}}}{9}, \frac{\overset{38}{\cancel{22}}}{9}\right)$$

distance from P to line = length of $QP - \text{proj}_{QR}(QP)$

$$= \left\| (2,-3,6) - \left(\frac{38}{9}, \frac{-19}{9}, \frac{38}{9}\right) \right\|$$

$$= \left\| \left(\frac{18-38}{9}, \frac{-27+19}{9}, \frac{54-38}{9}\right) \right\| = \left\| \left(\frac{-20}{9}, \frac{-8}{9}, \frac{16}{9}\right) \right\|$$

$$= \left\| \frac{4}{9}(-5,-2,4) \right\| = \frac{4}{9} \| (-5,-2,4) \| = \frac{4}{9}\sqrt{25+4+16}$$

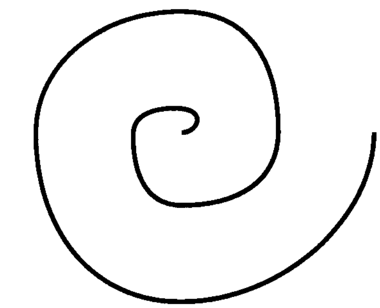
$$= \frac{4}{9}\sqrt{45} = \frac{4}{9}\sqrt{5 \cdot 9} = \frac{4}{9}\sqrt{5} \cdot \sqrt{9} = \frac{4}{9} \cdot \sqrt{5} \cdot 3 = \frac{4}{3}\sqrt{5}$$

8. (15 pts.) Find the arclength of the parametrized curve

$$\vec{r}(t) = (t^2 \cos t, t^2 \sin t) \qquad \text{for } 0 \le t \le 4\pi.$$

[Note: Take your time, the integral you get should not be horrible...]

Length $= \int_0^{4\pi} \|\vec{r}\,'(t)\|\, dt$



$$\vec{r}\,'(t) = (2t\cos t + t^2(-\sin t),\ 2t\sin t + t^2(\cos t))$$

$$= (2t\cos t - t^2\sin t,\ 2t\sin t + t^2\cos t)$$

$$\|\vec{r}\,'(t)\| = \left((2t\cos t - t^2\sin t)^2 + (2t\sin t + t^2\cos t)^2\right)^{1/2}$$

$$= \left(4t^2\cos^2 t \cancel{- 2t^3\sin t\cos t} + t^4\sin^2 t + 4t^2\sin^2 t \cancel{+ 2t^3\sin t\cos t} + t^4\cos^2 t\right)^{1/2}$$

$$= \left(4t^2\cos^2 t + 4t^2\sin^2 t + t^4\sin^2 t + t^4\cos^2 t\right)^{1/2}$$

$$= \left(4t^2(\cos^2 t + \sin^2 t) + t^4(\sin^2 t + \cos^2 t)\right)^{1/2} = (4t^2 + t^4)^{1/2}$$

$$= \left(t^2(4 + t^2)\right)^{1/2} = |t|(t^2+4)^{1/2} = t(t^2+4)^{1/2}$$

So Arclength $= \int_0^{4\pi} t(t^2+4)^{1/2}\, dt$ $\qquad u = t^2 + 4$, $du = 2t\, dt$ $\qquad t=0,\ u=4$; $t=4\pi,\ u=16\pi^2+4$

$$= \int_4^{16\pi^2+4} \frac{1}{2} u^{1/2}\, du = \frac{1}{2}\cdot\frac{2}{3} u^{3/2} \Big|_4^{16\pi^2+4} = \boxed{\frac{1}{3}\left((16\pi^2+4)^{3/2} - 4^{3/2}\right)}$$

$$= \frac{1}{3}\left(4^{3/2}(4\pi^2+1)^{3/2} - 4^{3/2}\right) = \frac{4^{3/2}}{3}\left((4\pi^2+1)^{3/2} - 1\right)$$

$$= \frac{8}{3}\left((4\pi^2+1)^{3/2} - 1\right)$$

Some possibly useful formulas

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\int \sec^n x \; dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \; dx$$

$$\int \sin^n x \; dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \; dx$$

$$\int \cos^n x \; dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \; dx$$

$$c^2 \int \frac{dy}{(y^2+c^2)^k} = \frac{1}{(2k-2)} \cdot \frac{y}{(y^2+c^2)^{k-1}} + \frac{(2k-3)}{(2k-2)} \int \frac{dy}{(y^2+c^2)^{k-1}}$$