How $\int \sec x \; dx = \ln|\sec x + \tan x| + C$ might have been discovered

As an illustration of the fact that there is often more than one way to discover an antiderivative, and a further illustration of the fact that one should not take the integrand at face value, here is another way to discover the antiderivative of $\sec x$, in a way that one might imagine the first discoverer would have found it:

$$\int \sec x \; dx = \int \frac{dx}{\cos x} = \int \frac{\cos x \; dx}{\cos^2 x} = \int \frac{\cos x \; dx}{1 - \sin^2 x}$$

Basically, effectively, the integrand has an odd number of cosines (-1 of them...), so our rule of thumb tells us to try $u = \sin x$. Now setting $u = \sin x$, so $du = \cos x \; dx$, then

$$\int \sec x \; dx = \int \frac{du}{1-u^2}\Big|_{u=\sin x} = \int \frac{du}{(1-u)(1+u)}\Big|_{u=\sin x}$$

But! $\dfrac{1}{(1-u)(1+u)} = \dfrac{1}{2}\Big(\dfrac{1}{1-u} + \dfrac{1}{1+u}\Big)$.

[Why? Just put the right-hand side back over a common denomenator. How might we have discovered this? This follows from the method of "partial fractions", which we will be discussing soon.] So:

$$\int \frac{du}{(1-u)(1+u)} = \int \frac{1}{2}\Big(\frac{1}{1-u} + \frac{1}{1+u}\Big)\; du = \frac{1}{2}\Big(\int \frac{du}{1-u} + \int \frac{du}{1+u}\Big)$$

Then setting $v = 1 - u$ (so $du = -dv$) in the first integral and setting $w = 1 + u$ (so $dw = du$) in the second, we have

$$\int \frac{du}{(1-u)(1+u)} = \frac{1}{2}\Big(-\int \frac{dv}{v}\Big|_{v=1-u} + \int \frac{dw}{w}\Big|_{w=1+u}\Big) = \frac{1}{2}\Big(-\ln|v|\Big|_{v=1-u} + \ln|w|\Big|_{w=1+u}\Big)$$
$$= \frac{1}{2}(-\ln|1-u| + \ln|1+u|) + C = \frac{1}{2}\ln\Big|\frac{1+u}{1-u}\Big| + C$$

So: $\displaystyle\int \sec x \; dx = \frac{1}{2}\ln\Big|\frac{1+u}{1-u}\Big| + C\Bigg|_{u=\sin x} = \frac{1}{2}\ln\Big|\frac{1+\sin x}{1-\sin x}\Big| + C$

Which looks nothing like our original answer! Except that

$$\frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{(1+\sin x)^2}{\cos^2 x}$$
$$= \Big(\frac{1+\sin x}{\cos x}\Big)^2 = \Big(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\Big)^2 = (\sec x + \tan x)^2,$$

so

$$\int \sec x \; dx = \frac{1}{2}\ln|(\sec x + \tan x)^2| + C = \frac{1}{2}\cdot 2\ln|\sec x + \tan x| + C$$
$$= \ln|\sec x + \tan x| + C,$$

as desired!