Math 107H Exam 1 Solutions

1. $\int_2^3 (3x-2)^{5/4}\,dx$ $\qquad u=3x-2,\ du=3dx,\ dx=\frac{1}{3}du$

$x=2 \to u=6-2=4,\ x=3 \to u=9-2=7$

$$=\int_4^7 u^{5/4}\left(\tfrac{1}{3}\right)du = \tfrac{1}{3}\left(\tfrac{4}{9}u^{\frac{9}{4}}\right)\Big|_4^7 = \tfrac{4}{27}\left(7^{9/4}-4^{9/4}\right).$$

[or: solve indefinite integral first, then plug in values.]

2. $\int x^{\frac{1}{3}}\ln x\,dx$ $\qquad u=\ln x \quad dv=x^{\frac{1}{3}}dx$

$du=\frac{1}{x}dx \quad v=\frac{3}{4}x^{4/3}$

$$=uv-\int v\,du = \tfrac{3}{4}x^{4/3}\ln x - \int \tfrac{3}{4}x^{4/3}\tfrac{1}{x}dx$$

$$=\tfrac{3}{4}x^{4/3}\ln x - \tfrac{3}{4}\int x^{1/3}dx = \tfrac{3}{4}x^{4/3}\ln x - \tfrac{3}{4}\left(\tfrac{3}{4}x^{4/3}\right)+C$$

$$=\tfrac{3}{4}x^{4/3}\ln x - \tfrac{9}{16}x^{4/3}+C.$$

[or: u-subs! $u=\ln x,\ du=\frac{1}{x}dx,\ dx=x\,du=e^u du$

$x^{1/3}=(e^u)^{\frac{1}{3}}=e^{u/3}$; gives $\int u e^{\frac{4}{3}u}du\Big|_{u=\ln x}$]

3. $$\int_0^{\frac{\pi}{2}}\cos^3x\,dx = \int_0^{\frac{\pi}{2}}(1-\sin^2x)(\cos x\,dx) = \int_0^{\frac{\pi}{2}}(1-\sin^2x)\cos x\,dx$$

$u=\sin x \quad du=\cos x\,dx \quad x=0\to u=0 \quad x=\frac{\pi}{2}\to u=1$

$$=\int_0^1(1-u^2)\,du = u-\frac{u^3}{3}\Big|_0^1 = \left(1-\tfrac{1}{3}\right)-(0-0)=\tfrac{2}{3}$$

4. $\displaystyle\int \frac{dx}{1+\sqrt{x}}$ $\quad u=\sqrt{x}=x^{\frac{1}{2}} \quad du=\frac{1}{2}x^{-\frac{1}{2}}dx=\frac{1}{2\sqrt{x}}dx$

$$=\int \frac{1}{1+\sqrt{x}}(2\sqrt{x})\left(\frac{1}{2\sqrt{x}}dx\right)=\int \frac{2\sqrt{x}}{1+\sqrt{x}}\left(\frac{1}{2\sqrt{x}}dx\right)=\int \frac{2u}{1+u}du\Big|_{u=\sqrt{x}}$$

$$=\int \frac{2(u+1)-2}{1+u}du\Big|_{u=\sqrt{x}}=\int 2-\frac{2}{u+1}du\Big|_{u=\sqrt{x}}=2u-2\ln|u+1|+C\Big|_{u=\sqrt{x}}$$

$$=2\sqrt{x}-2\ln|\sqrt{x}+1|+C.$$

[or: $u=1+\sqrt{x}$ works: $=\displaystyle\int \frac{2(u-1)}{u}du\Big|_{u=1+\sqrt{x}}=\ldots$]

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5. $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$: Set $x=\sin u$ $\quad dx=\cos u\,du$, so $\sqrt{1-x^2}=\cos u$

$$=\int \frac{\cos u\,du}{(\sin^2 u)\cos u}\Big|_{x=\sin u}=\int \frac{du}{\sin^2 u}\Big|_{x=\sin u}=\int \csc^2 u\,du\Big|_{x=\sin u}$$

$\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}dx$ Set $x=3\tan u$ $\quad dx=3\sec^2 u\,du$, so $\sqrt{x^2+9}=3\sec u$

$$=\int \frac{(3\tan u)^2}{3\sec u}(3\sec^2 u\,du)\Big|_{x=3\tan u}=9\int \tan^2 u\sec u\,du\Big|_{x=3\tan u}$$

6. $\dfrac{1}{(x+2)(x+5)} = \dfrac{A}{x+2} + \dfrac{B}{x+5} = \dfrac{A(x+5)+B(x+2)}{(x+2)(x+5)}$

so $1 = A(x+5)+B(x+2)$ $\quad x=-5: \ 1 = 0 + B(-3) \quad B = -1/3$

$\quad x=-2 \quad 1 = A(3)+0 \quad A = 1/3$

so $\dfrac{1}{(x+2)(x+5)} = \dfrac{1}{3}\dfrac{1}{x+2} - \dfrac{1}{3}\dfrac{1}{x+5}$

For $\dfrac{1}{(x^2+2)(x^2+5)}$, replace x with x^2! $\boxed{\dfrac{1}{3}\dfrac{1}{x^2+2} - \dfrac{1}{3}\dfrac{1}{x^2+5}}$.

so $\displaystyle\int \frac{dx}{(x^2+2)(x^2+5)} = \int \frac{1}{3}\frac{1}{x^2+2} - \frac{1}{3}\frac{1}{x^2+5}\,dx = \frac{1}{3}\int \frac{dx}{x^2+(\sqrt{2})^2} - \frac{1}{3}\int \frac{dx}{x^2+(\sqrt{5})^2}$

$= \dfrac{1}{3}\dfrac{1}{\sqrt{2}}\arctan\left(\dfrac{x}{\sqrt{2}}\right) - \dfrac{1}{3}\dfrac{1}{\sqrt{5}}\arctan\left(\dfrac{x}{\sqrt{5}}\right)$.

7. Error $\leq M \dfrac{(b-a)^3}{24n^2} < \dfrac{1}{100}$ $\quad b-a = 5-1 = 4$

$f(x) = \frac{1}{x}$, so $f'(x) = \dfrac{-1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$.

$f''(x)$ is decreasing on $[1,5]$ (its derivative is $\dfrac{-6}{x^4} < 0$)

so max of $f''(x)$ as at $x=1$: $M = 2$. So we want

$2\dfrac{(4)^3}{24n^2} < \dfrac{1}{100}$, so $\dfrac{2(4^3)100}{24} < n^2 = \cancel{\dfrac{128}{24}\cdot 100}$

so $n^2 > \dfrac{2\cdot 64}{24}\cdot 100 = \dfrac{64}{12}\cdot 100 = \dfrac{16}{3}\cdot 100$ so $n > \sqrt{\dfrac{16}{3}\cdot 100}$

$= \dfrac{\sqrt{16}}{\sqrt{3}}\cdot\sqrt{100} = \dfrac{4}{\sqrt{3}}\cdot 10$ $\quad \dfrac{4}{\sqrt{3}} \approx 2.4$? so $n > 2.4\cdot 10 = 24$ will work.

(bigger numbers will, too...)