115!

Solutions

Name:

Math 107H Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

1. (15 pts.) Use a comparision theorem to decide if the following improper integral converges (if yes, you do *not* need to find the value of the integral):

$$\int_7^\infty \frac{x \ln x}{x^2+1}\, dx$$

$\frac{x\ln x}{x^2+1}$ looks like $\frac{x\ln x}{x^2} = \frac{\ln x}{x}$ and

$$\int_7^\infty \frac{\ln x}{x}\,dx \underset{du=\frac{1}{x}dx}{\overset{u=\ln x}{=}} \int_{\ln 7}^\infty u\,du = \lim_{N\to\infty} \frac{u^2}{2}\Big|_{\ln 7}^{N} = \ln\frac{N^2}{2} - \frac{(\ln 7)^2}{2} = \infty$$

But! $\frac{x\ln x}{x^2+1} < \frac{x\ln x}{x^2} = \frac{\ln x}{x}$, so that comparison runs the wrong way.

But! $\frac{x\ln x}{x^2+1} > \frac{x\ln x}{x^2+x^2} = \frac{1}{2}\frac{\ln x}{x}$ (for $x>1$) and

$\int_7^\infty \frac{1}{2}\frac{\ln x}{x}\,dx = \infty$ (it's half of what we did above), so

$\int_7^\infty \frac{x\ln x}{x^2+1}\,dx$ diverges by comparison with $\int_7^\infty \frac{x\ln x}{2x^2}\,dx$.

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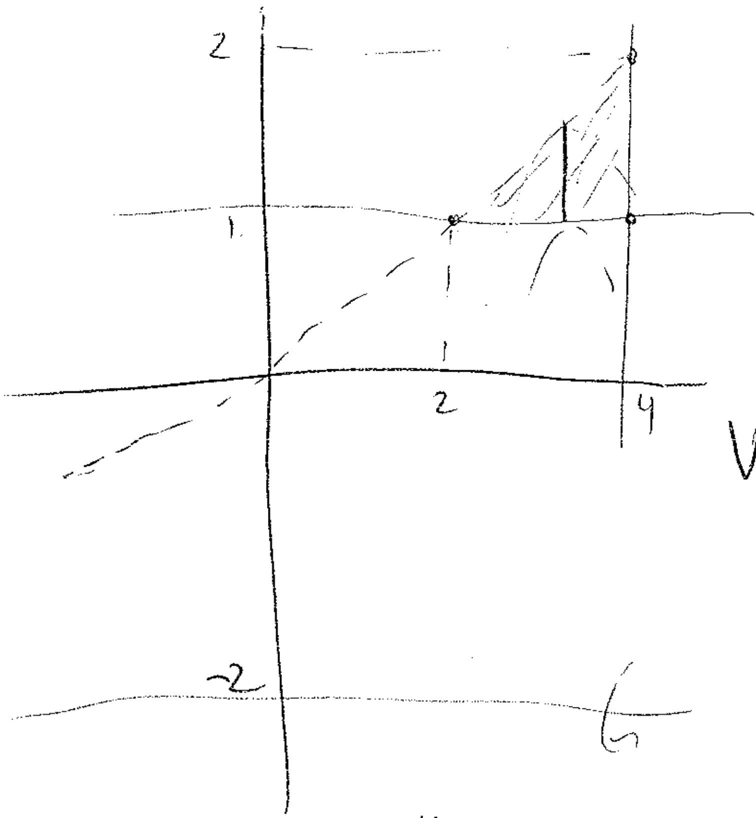
or: Use $\frac{x\ln x}{x^2+1} > \frac{x}{x^2+1}$ (for $x>e$) and that

$\int_7^\infty \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)\Big|_7^\infty$ to show divergence!

2. (20 pts.) Find the volume of the region obtained by spinning the triangle with sides lying along the lines $y = \frac{1}{2}x$, $x = 4$, and $y = 1$, *around the line* $y = -2$.



$y = 1 = \frac{1}{2}x \longrightarrow x = 2$

$R = \frac{1}{2}x - (-2) = \frac{1}{2}x + 2$

$r = 1 - (-2) = 1 + 2 = 3$

$$\text{Volume} = \int_2^4 \pi\left(\left(\tfrac{1}{2}x+2\right)^2 - (3)^2\right)dx$$

$$= \pi\int_2^4 \tfrac{1}{4}x^2 + 2x + 4 - 9 \; dx$$

$$= \pi\int_2^4 \tfrac{1}{4}x^2 + 2x - 5 \; dx = \pi\left(\tfrac{1}{12}x^3 + x^2 - 5x\right)\Big|_2^4$$

$$= \boxed{\pi\left(\left(\tfrac{64}{12} + 16 - 20\right) - \left(\tfrac{8}{12} + 4 - 10\right)\right)}$$

$$= \pi\left(\left(5 + \tfrac{1}{3} - 4\right) - \left(\tfrac{2}{3} - 6\right)\right) = \pi\left(\tfrac{4}{3} + \tfrac{16}{3}\right) = \tfrac{20}{3}\pi$$

1159

3. (15 pts.) Set up, *but do not evaluate*, the integral which evaluates to the length of the spiral, with parametric equation

$$x = t\cos t\ ,\ y = t\sin t\ , \qquad \text{for } 0 \le t \le 4\pi\ .$$

$x'(t) = \cos t - t\sin t \qquad y'(t) = \sin t + t\cos t$

$$\text{Length} = \boxed{\int_0^{4\pi} \left((\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 \right)^{1/2} dt}$$

This equals: $\cos^2 t - \cancel{2t\sin t\cos t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t$

$= (\cos^2 t + \sin^2 t) + t^2(\sin^2 t + \cos^2 t)$

$= 1 + t^2$, so

$$\text{Length} = \int_0^{4\pi} \sqrt{t^2+1}\, dt \text{ , which we can integrate!}$$

$\int \sqrt{t^2+1}\, dt \quad \begin{matrix} t = \tan u \\ dt = \sec^2 u\, du \end{matrix} \quad = \int \sqrt{\sec^2 u}\, \sec^2 u\, du \Big|_{t=\tan u} = \int \sec^3 u\, du \Big|_{t=\tan u}$

$\int \sec^3 u\, du = \int \sec u \sec^2 u\, du \quad \begin{matrix} w = \sec u \\ dw = \sec u\tan u\, du \\ dv = \sec^2 u\, du \\ v = \tan u \end{matrix} \quad = \sec u\tan u - \int \sec u\tan^2 u\, du$

$= \sec u\tan u - \int \sec^3 u - \sec u\, du$

So $2\int \sec^3 u\, du = \sec u\tan u + \int \sec u\, du = \sec u\tan u + \ln|\sec u + \tan u| + C$

So $\int \sqrt{t^2+1}\, dt = \frac{1}{2}(\sec u\tan u + \ln|\sec u + \tan u|)\Big|_{t=\tan u}$

$= \frac{1}{2} t\sqrt{t^2+1} + \frac{1}{2}\ln\left|\sqrt{t^2+1} + t\right|$

So $\text{Length} = \frac{1}{2}\left(t\sqrt{t^2+1} + \ln\left|\sqrt{t^2+1} + t\right| \right)\Big|_0^{4\pi} = \boxed{2\pi\sqrt{1+16\pi^2} + \frac{1}{2}\ln\left(4\pi + \sqrt{1+16\pi^2}\right)}$

1201

4. (10 pts. each) Find the limit of each of the following sequences, *if it exists*:

(a) $a_n = \dfrac{2+\sqrt{n^2+5n-1}}{7n+12}$

$$a_n = \frac{2+\sqrt{n^2+5n-1}}{7n+12}\cdot\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \frac{\frac{2}{n}+\sqrt{1+\frac{5}{n}-\frac{1}{n^2}}}{7+\frac{12}{n}}$$

$$\longrightarrow \frac{0+\sqrt{1+0-0}}{7+0} = \frac{\sqrt{1}}{7} = \boxed{\frac{1}{7}}$$

(b) $b_n = (n^2+2)^{\frac{1}{n}}$ [Hint: take logs, first!]

$$\ln(b_n) = \frac{1}{n}\ln(n^2+2) = \frac{\ln(n^2+2)}{n} = f(n) \text{ for}$$

$$f(x) = \frac{\ln(x^2+2)}{x} = \frac{g(x)}{h(x)}. \text{ Since } g(x), h(x) \to \infty \text{ as } x\to\infty$$

L'Hopital says

$$\lim_{x\to\infty} f(x) = \lim \frac{(\ln(x^2+2))'}{(x)'} = \lim \frac{\frac{2x}{x^2+2}}{1} = \lim \frac{2x}{x^2+2}$$

$$= \lim \frac{2/x}{1+2/x^2} = \frac{0}{1+0} = 0. \text{ So } \ln(b_n) \to 0$$

$$\text{as } n\to\infty, \text{ so } b_n = e^{\ln(b_n)} \longrightarrow e^0 = 1 \text{ as } n\to\infty$$

[Also, recall *L'Hopital's Rule*: if $f(x), g(x) \to \infty$ as $x \to \infty$, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}\ .]$$

1204

5. (15 pts. each) Decide whether or not each of the following series converges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^2}$

$\frac{\sqrt{n^2-1}}{n^2}$ looks like $\frac{\sqrt{n^2}}{n^2} = \frac{n}{n^2} = \frac{1}{n}$

and $\sum \frac{1}{n}$ diverges. Since

$$\frac{\frac{\sqrt{n^2-1}}{n^2}}{\frac{1}{n}} = \frac{\sqrt{n^2-1}}{n^2} \cdot \frac{n}{1} = \frac{\sqrt{n^2-1}}{n} = \frac{\sqrt{1-\frac{1}{n^2}}}{1} \longrightarrow \frac{\sqrt{1-0}}{1} = 1 \neq 0, \infty$$

and $\sum \frac{1}{n}$ diverges, $\sum \frac{\sqrt{n^2-1}}{n^2}$ diverges by limit comparison

1206

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{13n+1}{8n+7} \cdot \left(\frac{2}{3}\right)^n$

terms look like $\frac{13}{8}\left(\frac{2}{3}\right)^n$, and

$\sum \frac{13}{8}\left(\frac{2}{3}\right)^n = \frac{13}{8} \sum \left(\frac{2}{3}\right)^n$ converges.

Since
$$\frac{\frac{13n+1}{8n+7}\left(\frac{2}{3}\right)^n}{\frac{13}{8}\left(\frac{2}{3}\right)^n} = \frac{13n+1}{13} \cdot \frac{8}{8n+7} = \cancel{\frac{13+\frac{1}{n}}{13} \cdot 8}$$

$$= \frac{8\,(13+\frac{1}{n})}{13\,(8+\frac{7}{n})} \longrightarrow \frac{8\,(13+0)}{13(8+0)}$$

$$= \frac{8 \cdot 13}{13 \cdot 8} = 1 \neq 0, \infty \text{ and}$$

$\sum \left(\frac{13}{8}\right)\left(\frac{2}{3}\right)^n = \frac{13}{8} \sum \left(\frac{2}{3}\right)^n$

converges (geometric series, $a = \frac{2}{3} < 1$),

$\sum \frac{13n+1}{8n+7}\left(\frac{2}{3}\right)^n$ converges by limit comparison

1209