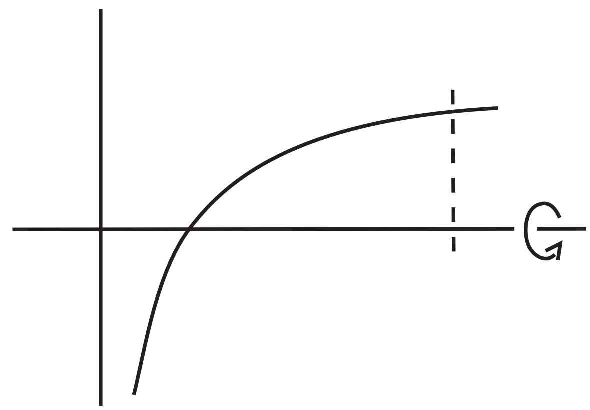
**Math 107H Practice Exam 2 Solutions**

**Note:** Most sequences/series can be shown to converge or diverge in more than one way; the solutions below illustrate only one such method. Your approach may differ....

1. Find the volume of the region obtained by revolving the region under the graph of $f(x) = \ln x$ from $x = 1$ to $x = 3$ around the $x$-axis (see figure).



Integrating slices $dx$: Volume$=\pi \int_1^3 (\ln x)^2\ dx = (*)$

By parts: $u = (\ln x)^2$ , $du = \dfrac{2\ln x}{x}\ dx$ , $dv = dx$ , $v = x$

$(*) = \pi x(\ln x)^2\Big|_1^3 - \pi \int_1^3 2\ln x\ dx = (**)$

By parts again: $u = 2\ln x$ , $du = \dfrac{2}{x}\ dx$ , $dv = dx$ , $v = x$

$(**) = \pi x(\ln x)^2\Big|_1^3 - \pi(2x\ln x|Big|_1^3 - \int_1^3 2\ dx) =$

noindent **2.** Find the improper integral $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\ dx$ .

$\displaystyle\int \frac{1}{x(\ln x)^3}\ dx = \int \frac{du}{u^3}\Big|_{u=\ln x}$ via the $u$-substitution $u = \ln x$, so $du = \dfrac{1}{x}\ dx$,

which equals $\displaystyle\int u^{-3}\ du\Big|_{u=\ln x} = -\frac{1}{2}u^{-2} + c\Big|_{u=\ln x} = -\frac{1}{2(\ln x)^2} + c$

So $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\ dx = \lim_{n\to\infty} \int_2^N \frac{1}{x(\ln x)^3}\ dx$

$\displaystyle = \lim_{n\to\infty} -\frac{1}{2(\ln x)^2}\Big|_2^N = \lim_{n\to\infty} \frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln N)^2}$

But since $\ln N \to \infty$ as $N \to \infty$, $\dfrac{1}{2(\ln N)^2} \to 0$ as $N \to \infty$, so

**3.** Determine the convergence or divergence of the following sequences:

**(a)** $a_n = \dfrac{n^3 + 6n^2\ln n - 1}{2 - 3n^3} = \dfrac{1 + 6(\ln n)/n - 1/n^3}{2/n^3 - 3}$.

and since $1/n^3 \to 0$ and $(\ln n)/n \to 0$ as $n \to \infty$,

$$a_n \to \frac{1+6\cdot 0-0}{2\cdot 0-3} = \frac{1}{-3} = -\frac{1}{3} \text{ as } n \to \infty.$$

**(b)** $b_n = \dfrac{n^{n+\frac{1}{n}}}{(n+3)^n}$

$$b_n = \frac{n^{n+\frac{1}{n}}}{(n+3)^n} = \frac{n^n n^{\frac{1}{n}}}{(n+3)^n} = \frac{n^{\frac{1}{n}}}{\left(\frac{n+3}{n}\right)^n} = \frac{n^{\frac{1}{n}}}{\left(1+\frac{3}{n}\right)^n}\,.$$

But $n^{\frac{1}{n}} \to 1$ and $\left(1+\dfrac{3}{n}\right)^n \to e^3$ as $n \to \infty$, so $b_n \to \dfrac{1}{e^3} = e^{-3}$ as $n \to \infty$ .

**4.** Determine the convergence or divergence of the following series:

**(a)** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)(\ln n)^{2/3}}$ [Hint: limit compare, then integral...]

$a_n = \dfrac{1}{(n-1)(\ln n)^{2/3}}$ looks like $b_n = \dfrac{1}{n(\ln n)^{2/3}}$, and $\dfrac{a_n}{b_n} = \dfrac{n}{n-1} \to 1$ as $n \to \infty$,

so $\sum a_n$ converges precisely when $\sum b_n$ converges. But:

$b_n = \dfrac{1}{n(\ln n)^{2/3}} = f(n)$ for $f(x) = \dfrac{1}{x(\ln x)^{2/3}}$, which is continuous and decreasing ($x$ and $\ln(x)$ are both increasing, so $(\ln x)^{2/3}$ is increasing, so their reciprocals are decreasing, and so the product is decreasing). So we can apply the integral test:

$\displaystyle\int \frac{1}{x(\ln x)^{2/3}}\,dx = \int \frac{du}{u^{2/3}}\,du|_{u=\ln x} = 3u^{1/3}|_{u=\ln x} = 3(\ln x)^{1/3}$, so

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\,dx = \lim_{N\to\infty}[3(\ln N)^{1/3} - 3(\ln 2)^{1/3}]$, but since $\ln N \to \infty$ as $N \to \infty$,

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\,dx$ diverges, so $\sum b_n$ diverges, so $\sum a_n$ **diverges**.

**(b)** $\displaystyle\sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ $a_n = \dfrac{6n}{(1-n^2)^2}$ looks like $b_n = \dfrac{6n}{(-n^2)^2} = \dfrac{6}{n^3}$, which converges:

So note that $\dfrac{a_n}{b_n} = \dfrac{(-n^2)^2}{(1-n^2)^2} = \dfrac{1}{(\frac{1}{n^2}-1)^2}$, and since $1/n^2 \to 0$ as $n \to \infty$, $\dfrac{a_n}{b_n} \to \dfrac{1}{(0-1)^2} = 1$ as $n \to \infty$, so by limit comparison, $\sum a_n$ converges precisely when $\sum b_n$ converges.

But: $\sum b_n = \sum \dfrac{6}{n^3} = 6\sum \dfrac{1}{n^3}$, which converges ($p$-series, $p = 3 > 1$), so $\sum b_n$ converges, so $\sum a_n = \displaystyle\sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ **converges**.

**6.** Set up, **but do not evaluate**, the integral which will compute the arclength of the graph of $y = x\sqrt{1+x^2}$ from $x = 0$ to $x = 3$.

$f(x) = x\sqrt{1+x^2} = x(1+x^2)^{\frac{1}{2}}$, so $f'(x) = (1+x^2)^{\frac{1}{2}} + x(\frac{1}{2})(1+x^2)^{-\frac{1}{2}}(2x) = (1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}$.

So Arclength $= \int_0^3 \sqrt{1+[f'(x)]^2}\, dx = \int_0^3 \sqrt{1+[(1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}]^2}\, dx$

6. Find the following limits:

(a) $\lim_{n\to\infty} \frac{1+\sqrt{2n}}{\sqrt{n}} = (*)$

$$(*) = \lim_{n\to\infty} \frac{(1/\sqrt{n}) + (\sqrt{2}\sqrt{n}/\sqrt{n})}{(\sqrt{n}/\sqrt{n})} = \lim_{n\to\infty} \frac{(1/\sqrt{n}) + \sqrt{2}}{1}$$
$$= \lim_{n\to\infty} \sqrt{1/n} + \sqrt{2} = \sqrt{0} + \sqrt{2} = \sqrt{2},$$

since $a_n = 1/n \to 0$ as $n \to \infty$, so $\sqrt{a_n} \to \sqrt{0}$, since $\sqrt{x} \to 0$ as $x \to 0$ .

(b) $\lim_{n\to\infty} \frac{4^n + 3^n}{4^n - 3^n} = (**)$

$$(**) = \lim_{n\to\infty} \frac{4^n/4^n + 3^n/4^n}{4^n/4^n - 3^n/4^n} = \lim_{n\to\infty} \frac{1+(3/4)^n}{1-(3/4)^n} = \frac{1+0}{1-0} = 1,$$

since $(3/4)^n \to 0$ as $n \to \infty$, since $|3/4| < 1$ .

8. *Use a comparison test* to determine the convergence or divergence of each of the following series:

(a) $\sum_{n=0}^{\infty} \frac{n^{\frac{1}{3}}}{\sqrt{n^3+7}}$

Looking at the dominant terms, this series behaves like one with $n$-th term $\frac{n^{\frac{1}{3}}}{\sqrt{n^3}} = n^{\frac{1}{3}-\frac{3}{2}} = n^{-\frac{7}{6}}$, which converges.

More precisely, $\lim_{n\to\infty} \frac{\frac{n^{\frac{1}{3}}}{\sqrt{n^3+7}}}{\frac{n^{\frac{1}{3}}}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^3+7}{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{7}{n^3}}} = \frac{1}{\sqrt{1+0}} = 1.$

So since $\sum_{n=1}^{\infty} n^{-\frac{7}{6}}$ converges [$p$-series with $p = \frac{7}{6} > 1$],

$$\sum_{n=0}^{\infty} \frac{n^{\frac{1}{3}}}{\sqrt{n^3+7}} \text{ converges, by the limit comparison test.}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$

Looking at the dominant terms, this series behaves like one with $n$-th term $\dfrac{2^n}{n^2 2^n} = \dfrac{1}{n^2}$, which converges.

More precisely, $\displaystyle\lim_{n\to\infty} \frac{\left(\frac{n+2^n}{n^2 2^n}\right)}{\left(\frac{2^n}{n^2 2^n}\right)} = \lim_{n\to\infty} \frac{\frac{n}{2^n}+1}{1} = \lim_{n\to\infty} \frac{n}{2^n} + 1 = 0+1 = 1$, since

$\dfrac{n}{2^n} \to 0$ as $n \to \infty$, by L'Hôpital:

$\displaystyle\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{(x)'}{(2^x)'} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$, since $2^x$ gets (really) large as $x$ gets large.

So since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges [$p$-series with $p = 2 > 1$],

$$\sum_{n=0}^{\infty} \frac{n+2^n}{n^2 2^n} \text{ converges, by the limit comparison test.}$$