**Math 107H**

**Topics for the second exam**

**Technically, everything for the first exam! Plus:**

## Improper integrals

Fund Thm of Calc: $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, where $F'(x) = f(x)$

Problems: $a = -\infty$, $b = \infty$; $f$ blows up at $a$ or $b$ or somewhere in between integral is"improper"; usual technique doesn't work. Solution to this:

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x \qquad \int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x$$

(blow up at $a$) $\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to a^+}\int_r^b f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b f(x)\,\mathrm{d}x$

(similarly for blowup at $b$ (or both!))

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{s\to b^-}\int_a^s f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+}\int_a^{b-\epsilon} f(x)\,\mathrm{d}x$$

(blows up at $c$ (b/w $a$ and $b$)) $\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to c^-}\int_a^r f(x)\,\mathrm{d}x + \lim_{s\to c^+}\int_s^b f(x)\,\mathrm{d}x$

The integral <u>converges</u> if (all of the) limit(s) are finite

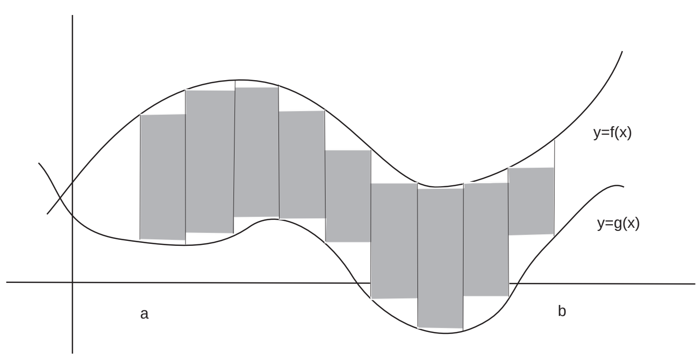
Comparison: $0 \le f(x) \le g(x)$ for all $x$;

if $\int_a^\infty g(x)\,\mathrm{d}x$ *converges*, so does $\int_a^\infty f(x)\,\mathrm{d}x$ ;

if $\int_a^\infty f(x)\,\mathrm{d}x$ *diverges*, so does $\int_a^\infty g(x)\,\mathrm{d}x$ .

## Applications of integration

**Area between curves.** Region between two curves; approximate by rectangles




Area $= \int_{left}^{right} (top) - (bottom)\; dx = \int_a^b f(x) - g(x)\; dx$

Integrate $dy$: Area $= \int_{bottom}^{top} (right) - (left)\; dy$

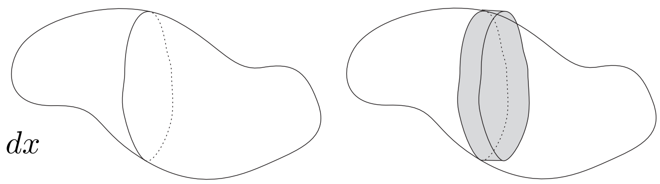
Which function is top/bottom changes? Cut interval into pieces, and use $\int_a^b = \int_a^c + \int_c^b$

Sometimes to calculate area between $f(x)$ and $g(x)$, need to first figure out limits of integration; solve $f(x) = g(x)$, then decide which one is bigger in between each pair of solutions.

**Volume by slicing.** To calculate volume, aprroximate region by objects whose volume we can calculate.

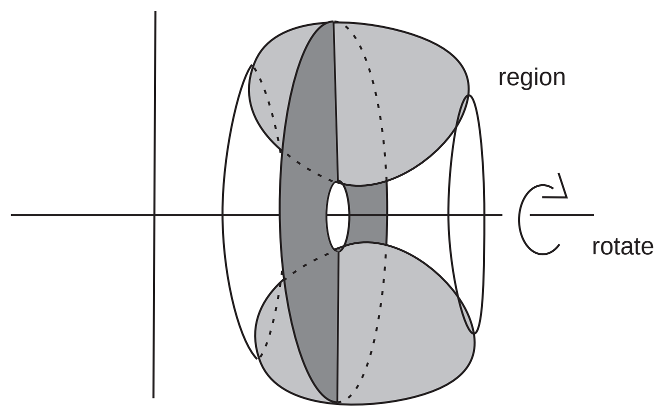
Volume $\approx \sum$(volumes of 'cylinders')
$= \sum$(area of base)(height)
$= \sum$(area of cross-section)$\Delta x_i$ .

So volume $= \int_{left}^{right}$ (area of cross section) $dx$



**Solids of revolution: disks and washers.** Solid of revolution: take a region in the plane and revolve it around an axis in the plane.

take cross-sections perpendicular to axis of revolution;
cross-section = disk (area=$\pi r^2$)
or washer (area=$\pi R^2 - \pi r^2$)
rotate around $x$-axis: write $r$ (and $R$) as functions of $x$, integrate $dx$
rotate around $y$-axis: write $r$ (and $R$) as functions of $y$, integrate $dy$




Otherwise, everything is as before: volume $= \int_{left}^{right} A(x)\, dx$ or volume $= \int_{bottom}^{top} A(y)\, dy$

The same is true if axis is parallel to $x-$ or $y-$axis; $r$ and $R$ just change (we add a constant).

**Arclength.** Idea: approximate a curve by lots of short line segments; length of curve $\approx$ sum of lengths of line segments.

A parametric curve is the path traced out by a point moving in the plane. To describe its position at time $t$, we need to know its coordinates: $x = x(t), y = y(t)$. Line segment between $(x(t_i), y(t_i))$ and $(x(t_{i+1}), y(t_{i+1}))$ has length

$$\sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2}$$

$$= \sqrt{[\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}]^2 + [\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}]^2} \cdot (t_{i+1} - t_i) \approx \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2} \cdot \Delta t_i$$

So length of curve $= \int_{\text{start}}^{\text{stop}} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

The problem: integrating $\sqrt{[x'(t)]^2 + [y'(t)]^2}$ ! Sometimes, $[x'(t)]^2 + [y'(t)]^2$ turns out to be a perfect square.....

Special case: curve is the graph of a function $y = f(x)$. Parametrize: $x = t$, $y = f(t)$, so

$$\text{length of curve} = \int_{\text{left}}^{\text{right}} \sqrt{1 + [f'(t)]^2}\ dt = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

**Work.** In physics, one studies the behavior of objects when acted upon by various *forces*. Newton's Laws provide the basic connection between a force acting on an object and the effect it has on its motion:

$$F = ma\ ; \qquad \text{Force} = \text{mass} \times \text{acceleration}$$

In physics, *work* represents force being applied across a distance. If a constant force $F$ is applied to an object, which moves the object a distance $D$, then the work done on the object is $W = F \cdot D$ . Again, if the force applied across this distance is not constant, then we interpret work, in stead, as an integral, by cutting the distance covered into small pieces of length $\delta x$ :

$$W \approx \sum F(x_i)\ \Delta x \text{ , so } W = \int_0^D F(x)\ dx$$

An interesting application of these ideas comes when trying to compute the amount of work necessary to pump out a tank of some known shape. If the tank has height D (we will think of the top of the tank as being at $x = 0$ and the bottom being at $x = D$), and at height $X$ our cross-section of the tank has area $A(x)$, then if (as when we computed volume) we think of the fluid in the tank as being a stack of cylinders with height $\Delta x$, the work necessary to lift the slice at height $x$ to the top of the tank will be

$$W = \text{(force)(distance)} = (m \cdot g) \cdot x = ((A(x) \cdot \Delta x)\rho g) \cdot x$$

where $\rho$ is the density of the fluid, $m$ = mass = (volume)(density), and $g$ is the accelration due to gravity (which is the force we need to overcome to push the fluid up out of the tank). Therefore, the work done to empty the tank is approximated by a sum of such quantities, which in turn models a definite integral; the work done in emptying the tank is

$$W = \rho g \int_0^D xA(x)\ dx$$

**Infinite sequences and series**

**Limits of sequences of numbers**

A sequence is: a string of numbers; a function $f$:$\mathbf{N} \to \mathbf{R}$; write $f(n) = a_n$

$a_n$ = $n$-th term of the sequence

Basic question: convergence/divergence: $\lim_{n\to\infty} a_n = L$ (or $a_n \to L$) if

eventually all of the $a_n$ are always as close to $L$ as we like, i.e.

for any $\epsilon > 0$, there is an $N$ so that if $n \geq N$ then $|a_n - L| < \epsilon$

Ex.: $a_n = 1/n$ converges to 0 ; can always choose $N$=$1/\epsilon$

$a_n = (-1)^n$ diverges; terms of the sequence never settle down to a <u>single</u> number

If $a_n = f(n)$ for $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{x\to\infty} f(x) = L$, then $a_n \to L$ as $n \to \infty$

(allows us to use L'Hôpital's Rule!)

If $a_n$ is increasing ($a_{n+1} \geq a_n$ for every $n$) and bounded from above

($a_n \leq M$ for every $n$, for some $M$) , then $a_n$ converges (but not necessarily to $M$ !)

limit is smallest number bigger than all of the terms of the sequence

**Limit theorems for sequences**

Idea: limits of sequences are a lot like limits of functions

If $a_n \to L$ and $b_n \to M$, then

$(a_n + b_n \to L + M \qquad (a_n - b_n) \to L - M \qquad (a_n b_n) \to LM$ , and

$(a_n/b_n) \to L/M$ (provided $M$, all $b_n$ are $\neq 0$)

Squeeze play theorem: if $a_n \leq b_n \leq c_n$ (for all $n$ large enough) and $a_n \to L$ and $c_n \to L$ , then $b_n \to L$

If $a_n \to L$ and $f$:$\mathbf{R}\to\mathbf{R}$ is continuous at $L$, then $f(a_n) \to f(L)$

Another basic list: ($x$ = fixed number, $k$ = konstant)

$$\frac{1}{n} \to 0 \qquad k \to k \qquad x^{\frac{1}{n}} \to 1$$

$$n^{\frac{1}{n}} \to 1 \qquad (1+\frac{x}{n})^n \to e^x \qquad \frac{x^n}{n!} \to 0$$

$x^n \to \{$ 0, if $|x| < 1$ ; 1, if $x = 1$ ; diverges, otherwise $\}$

**Infinite series**

An infinite series is an infinite sum of numbers

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n \qquad \text{(summation notation)}$$

$n$-th term of series = $a_n$ ; $N$-th partial sum of series = $s_N = \sum_{n=1}^{N} a_n$

An infinite series **converges** if the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges

We may start the series anywhere: $\sum_{n=0}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=3437}^{\infty} a_n$, etc. ;

convergence is unaffected (but the number it adds up to is!)

Ex. geometric series: $\quad a_n = ar^n$ ; $\quad \sum_{k=0}^{n} a_k = a\frac{r^{n+1}-1}{r-1}$ ; $\quad \sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$

**if** $|r| < 1$; otherwise, the series **diverges**.

Application: compound interest. Principal $P$ earning interest rate $r$ each time period, then amount accumulated after $n$ time periods is

$$(1+r)^n P = (1+r)(1+r)^{n-1}P = (1+r)^{n-1}P + r(1+r)^{n-1}P$$

= (amount in account at time $n-1$) + (interest earned in $n$-th time interval).

If $P$ is deposited <u>each</u> time period, then amount after $n$ is

$$P\frac{(1+r)^{n+1}-1}{(1+r)-1} = \sum_{k=0}^{n}(1+r)^k P = P + (1+r)\textstyle\sum_{k=0}^{n-1}(1+r)^k P$$

= (deposit at time $n$) + (amount in account at time $n-1$)

+ (interest earned on amount present at time $n-1$)

Ex. Telescoping series: partial sums $s_N$ 'collapse' to a compact expression

E.g. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \sum_{n=1}^{\infty}\frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})$; $s_N = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - (\frac{1}{N+1} + \frac{1}{N+2}))$

$n$-th term test: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

So if the $n$-th terms **don't** go to 0, then $\sum_{n=1}^{\infty} a_n$ diverges

Basic limit theorems: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

$\sum_{n=1}^{\infty}(a_n + b_n)= \sum_{n=1}^{\infty} a_n+\sum_{n=1}^{\infty} b_n$ $\qquad$ $\sum_{n=1}^{\infty}(a_n - b_n)= \sum_{n=1}^{\infty} a_n-\sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty}(ka_n)= k\sum_{n=1}^{\infty} a_n$

Truncating a series: $\sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \sum_{n=1}^{N-1} a_n$

**The integral test**

Idea: $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ all $n$, then the partial sums $\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence;

so converges exactly when bounded from above

**If** (eventually) $a_n = f(n)$ for a **decreasing** function $f : [a, \infty) \to \mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x)\ \mathrm{d}x \leq s_N = \sum_{n=a}^{N} a_n \leq \int_a^N f(x)\ \mathrm{d}x$$

so $\sum_{n=a}^{\infty} a_n$ converges exactly when $\int_a^{\infty} f(x)\ \mathrm{d}x$ converges

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$ $\qquad$ ($p$-series)

Ex: $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges exactly when $p > 1$ $\qquad$ (logarithmic $p$-series?)

These families of series make good test cases for comparison with more involved terms (see below!)

**Comparison tests**

Again, think $\sum_{n=1}^{\infty} a_n$ , with $a_n \geq 0$ all $n$

Convergence depends only on partial sums $s_N$ being **bounded**

One way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all $n$ (past a certain point), then

if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$ ; if $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: $a_n$ and $b_n \geq 0$ for all $n$, $\dfrac{a_n}{b_n} \to L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ anf $\sum b_n$ either **both** converge or **both** diverge

If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$

If $L = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$

(Why? eventually $(L/2)b_n \leq a_n \leq (3L/2)b_n$ ; so can use comparison test.)

Ex: $\sum 1/(n^3 - 1)$ converges; L-comp with $\sum 1/n^3$

$\sum n/3^n$ converges; L-comp with $\sum 1/2^n$

$\sum 1/(n \ln(n^2 + 1))$ diverges; L-comp with $\sum 1/(n \ln n)$