A long arclength problem

Arclength of $f(x) = e^x$ from a to b:

For notational convenience, we will compute the needed antiderivative first.
$f(x) = e^x$, so $f'(x) = e^x$, so $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + (e^x)^2}$.
We compute $\int \sqrt{1 + (e^x)^2}\ dx$:

Setting $u = e^x$, $du = e^x\ dx$ so $dx = \dfrac{du}{e^x} = \dfrac{du}{u}$, so

$$\int \sqrt{1 + (e^x)^2}\ dx = \int \frac{\sqrt{1 + u^2}}{u}\ du|_{u=e^x}\ .$$

Setting $u = \tan v$, we have $du = \sec^2 v\ dv$ and $\sqrt{1 + u^2} = \sec v$, so

$$\begin{aligned}\int \frac{\sqrt{1+u^2}}{u}\ du|_{u=e^x} &= \int \frac{\sec v}{\tan v} \sec^2 v\ dv|_{u=\tan v}|_{u=e^x} = \int \frac{\sec^3 v}{\tan v}\ dv|_{u=\tan v}|_{u=e^x} \\ &= \int \frac{1}{\sin v \cos^2 v}\ dv|_{u=\tan v}|_{u=e^x} = \int \frac{\sin v}{\sin^2 v \cos^2 v}\ dv|_{u=\tan v}|_{u=e^x} \\ &= \int \frac{\sin v}{(1 - \cos^2 v)\cos^2 v}\ dv|_{u=\tan v}|_{u=e^x}\end{aligned}$$

Setting $w = \cos v$, we have $dw = -\sin v\ dv$, so

$$\int \frac{\sin v}{(1 - \cos^2 v)\cos^2 v}\ dv|_{u=\tan v}|_{u=e^x} = \int \frac{-dw}{(1 - w^2)w^2}|_{w=\cos v}|_{u=\tan v}|_{u=e^x}$$

But

$$\begin{aligned}\frac{-1}{(1 - w^2)w^2} &= \frac{1}{w^2(w - 1)(w + 1)} = \frac{A}{w} + \frac{B}{w^2} + \frac{C}{w - 1} + \frac{D}{w + 1} \\ &= \frac{A(w(w - 1)(w + 1) + B(w - 1)(w + 1) + Cw^2(w + 1) + Dw^2(w - 1)}{w^2(w - 1)(w + 1)}, \text{ so}\end{aligned}$$

$1 = A(w(w - 1)(w + 1) + B(w - 1)(w + 1) + Cw^2(w + 1) + Dw^2(w - 1)$.

Setting $x = 0$, we get $1 = B(-1)$, so $B = -1$.
Setting $x = 1$, we get $1 = C(1)^2(2)$, so $C = 1/2$.
Setting $x = -1$, we get $1 = D(-1)^2(-2)$, so $D = -1/2$.

The last unknown coefficient, A, we can obtain by picking any other value for x; e.g., $x = 2$ gives
$1 = 6A + 3B + 12C + 4D = 6A - 3 + 6 - 2 = 6A + 1$, so $6A = 0$ so $A = 0$. So:

$$\frac{-1}{(1 - w^2)w^2} = -\frac{1}{w^2} + \frac{1}{2}\frac{1}{w - 1} - \frac{1}{2}\frac{1}{w + 1}\ .\ \text{So:}$$

$$\begin{aligned}\int \sqrt{1 + (e^x)^2}\ dx &= \int \frac{-1}{w^2} + \frac{1}{2}\frac{1}{w - 1} - \frac{1}{2}\frac{1}{w + 1}\ dw|_{w=\cos v}|_{u=\tan v}|_{u=e^x} \\ &= \frac{1}{w} + \frac{1}{2}\ln|w - 1| - \frac{1}{2}\ln|w + 1| + c|_{w=\cos v}|_{u=\tan v}|_{u=e^x}\end{aligned}$$

If $w = \cos v$ and $u = \tan v$, then $w = (u^2 + 1)^{-\frac{1}{2}}$.
And if $u = e^x$, then $w = ((e^x)^2 + 1)^{-\frac{1}{2}} = (e^{2x} + 1)^{-\frac{1}{2}}$. So

$$\begin{aligned}\int \sqrt{1 + e^{2x}}\ dx &= \frac{1}{w} + \frac{1}{2}\ln|w - 1| - \frac{1}{2}\ln|w + 1| + c|_{w=(e^{2x}+1)^{-\frac{1}{2}}} \\ &= (1 + e^{2x})^{\frac{1}{2}} + \frac{1}{2}\ln|(e^{2x} + 1)^{-\frac{1}{2}} - 1| - \frac{1}{2}\ln|(e^{2x} + 1)^{-\frac{1}{2}} + 1| + c, \text{ so}\end{aligned}$$

$$\begin{aligned}\int_a^b \sqrt{1 + e^{2x}}\ dx &= (1 + e^{2x})^{\frac{1}{2}} + \frac{1}{2}\ln|(e^{2x} + 1)^{-\frac{1}{2}} - 1| - \frac{1}{2}\ln|(e^{2x} + 1)^{-\frac{1}{2}} + 1||_a^b \\ &= \sqrt{1 + e^{2b}} - \sqrt{1 + e^{2a}} + \frac{1}{2}\ln\left|\frac{(\sqrt{1 + e^{2b}} - 1)(\sqrt{1 + e^{2a}} + 1)}{(\sqrt{1 + e^{2a}} - 1)(\sqrt{1 + e^{2b}} + 1)}\right|\end{aligned}$$