Why Lagrange was Right

We computed $\int_0^\pi \sin x \; dx = 2$ directly from the limit of Riemann sums definition, with the help of what is known as (one of) *Lagrange's Trigonometric Identity*(ies):

$$\sum_{i=1}^{n} \sin(ix) = \frac{1}{2}\cot(\frac{x}{2}) - \frac{\cos(n+\frac{1}{2})x}{2\sin(\frac{x}{2})}$$

We can show that this formula is correct (essentially finishing this integral computation): Noting first that the right-hand side of Lagrange's identity is equal to $\dfrac{\cos(\frac{x}{2}) - \cos(n+\frac{1}{2})x}{2\sin(\frac{x}{2})}$ and setting $u = \dfrac{x}{2}$ (so $x = 2u$), since all of those denominators are annoying,
what Lagrange really says is that

$$(*) \quad \sum_{i=1}^{n} \sin(2iu) = \frac{\cos(u) - \cos(2n+1)u}{2\sin(u)}$$

We can show that this is true, using a technique known as *mathematical induction*: we show that (*) holds when $n = 1$, and <u>if</u> (*) holds when $n = N$ then it <u>also</u> holds when $n = N + 1$. So being true for $n = 1$ means it is true for $n = 2$, so it is true for $n = 3$, and so for $n = 4$, and so on and so on!

For starters, we need that (for $n = 1$)

$$\sin(2u) = \frac{\cos(u) - \cos(3u)}{2\sin(u)}, \text{ that is, } 2\sin(u)\sin(2u) = \cos(u) - \cos(3u)$$

But this is a kind of trig identity you might know: setting $A = u$ and $B = 2u$, then using angle sum and difference formulas for cosine, we have

$$\begin{aligned}\cos(u) - \cos(3u) &= \cos(B-A) - \cos(B+A)\\ &= [\cos(B)\cos(A) + \sin(B)\sin(A)] - [\cos(B)\cos(A) - \sin(B)\sin(A)]\\ &= 2\sin(B)\sin(A) = 2\sin(A)\sin(B) = 2\sin(u)\sin(2u)\end{aligned}$$

so (*) is true when $n = 1$. We should remember this fact:

$$(**) \; 2\sin(A)\sin(B) = \cos(B-A) - \cos(B+A)$$

since we will use it again shortly! Now suppose that

$$\sum_{i=1}^{N} \sin(2iu) = \frac{\cos(u) - \cos(2N+1)u}{2\sin(u)}$$

Then:

$$\sum_{i=1}^{N+1} \sin(2iu) = \left\{\sum_{i=1}^{N} \sin(2iu)\right\} + \sin(2(N+1)u) = \left\{\frac{\cos(u) - \cos(2N+1)u}{2\sin(u)}\right\} + \sin(2N+2)u$$

$$= \frac{\cos(u) - [\cos(2N+1)u - 2\sin(u)\sin(2N+2)u]}{2\sin(u)}$$

What we want this to be equal to is

$$\frac{\cos(u)-\cos(2(N+1)+1)u}{2\sin(u)}=\frac{\cos(u)-\cos(2N+3)u}{2\sin(u)}$$

Comparing the two expressions, to make them equal what we need is that

$$\cos(2N+3)u=\cos(2N+1)u-2\sin(u)\sin(2N+2)u,$$

that is

$$2\sin(u)\sin(2N+2)u=\cos(2N+1)u-\cos(2N+3)u$$

But that might look kind of familiar! It sounds a lot like the first identity that we established. And, in fact, if we set $A=u$ and $B=(2N+2)u$, this equality that we need reads

$2\sin(A)\sin(B)=\cos(B-A)-\cos(B+A)$,

which is the trig identity that we already established! So we have shown that if (*) holds when $n=N$ then it also holds when $n=N+1$. So our "inductive" step holds, and we have established that (*) holds for every $n=1,2,3,4,\ldots$. Which is what we needed to establish that $\int_0^\pi \sin x\ dx=2$ without appealing to the Fundamental Theorem of Calculus....

In the end, once we've seen why it holds, perhaps the 'right' way to view Lagrange's identity is by clearing the denominator:

$$\sum_{i=1}^{n}2\sin(u)\sin(2iu)=2\sin(u)\sin(2u)+2\sin(u)\sin(4u)+\cdots+2\sin(u)\sin(2nu)$$

$$=[\cos(u)-\cos(3u)]+[\cos(3u)-\cos(5u)]+[\cos(5u)-\cos(7u)]+\cdots+[\cos((2n-1)u)-\cos((2n+1)u)]$$

$$=\cos(u)-[\cos(3u)-\cos(3u)]-[\cos(5u)-\cos(5u)]-[\cos(7u)-\cdots-\cos((2n-1)u)]-\cos((2n+1)u)$$

$$=\cos(u)-0-0-\cdots-0-\cos(2n+1)u=\cos(u)-\cos(2n+1)u\ .$$

The sum, once we have used the trig identity (**), becomes what we will call a 'telescoping' sum; when we rearrange the terms it collapses on itself. This is a concept we will return to (and exploit) in other situations in the future!

This same trig identity can be used to show that $\int_0^{\text{anything}} \sin x\ dx=1-\cos(\text{anything})$, by replacing x in the identity with the appropriate replacement for $\pi/(2n)$ (namely, anything$/(2n)$). This then leads to $\int_{\text{blah}}^{\text{bleh}} \sin x\ dx=\cos(\text{bleh})-\cos(\text{blah})$ (by 'splitting' the interval [blah, bleh] at 0), which is our familiar answer to this integration problem!