Math 107H, Section 1

Quiz number 3 solution

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Use trigonometric substitution to find the following antiderivative:

$$\int \frac{\sqrt{4-x^2}}{x}\,dx$$

The form inside the square root indicates that we should try $x = 2\sin u$,
so $dx = 2\cos u\,du$ and $\sqrt{4-x^2} = \sqrt{4-4\sin^2 u} = \sqrt{4}\cdot\sqrt{1-\sin^2 u} = 2\cos u$.
So:

$$\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{(2\cos u)(2\cos u\,du)}{2\sin u}\Big|_{x=2\sin u} = \int \frac{(4\cos^2\,du}{2\sin u}\Big|_{x=2\sin u}$$
$$= 2\int \frac{\cos^2 u}{\sin u}\,du\Big|_{x=2\sin u} = 2\int \frac{1-\sin^2 u}{\sin u}\,du\Big|_{x=2\sin u}$$
$$= 2\int \csc u - \sin u\,du\Big|_{x=2\sin u} = 2(-\ln|\csc u + \cot u| + \cos u)\Big|_{x=2\sin u}$$

Since $x = 2\sin u$, we have $\sin u = \frac{x}{2}$, so $\cos u = \frac{\sqrt{4-x^2}}{2}$, and then

$$\csc u = \frac{2}{x} \text{ and } \cot u = \frac{\cos u}{\sin u} = \frac{\sqrt{4-x^2}}{x}.$$

So:

$$\int \frac{\sqrt{4-x^2}}{x}\,dx = 2(-\ln|\csc u + \cot u| + \cos u)\Big|_{x=2\sin u}$$
$$= -2\ln\left|\frac{2}{x} + \frac{\sqrt{4-x^2}}{x}\right| + 2\frac{\sqrt{4-x^2}}{2} + c$$

This can be cleaned up a bit; it equals $\quad 2\ln|x| - 2\ln|\sqrt{4-x^2}+2| + \sqrt{4-x^2} + c$

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