How $\int \sec x \; dx$ <u>could</u> have been discovered

The fact that $\int \sec x \; dx = \ln|\sec x + \tan x| + C$ is usually presented by noting that

$$\int \sec x \; dx = \int \frac{(\sec x + \tan x)\sec x}{\sec x + \tan x} \; dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \; dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

This always seems rather artificial (although, ultimately, perhaps memorable, and a good lesson that we should allow ourselves to multiply an integrand by 1 in imaginative ways!).

But how might this integral <u>first</u> have been discovered? There is in fact a somewhat ugly way to get there, which has the advantage that it is in some sense "inevitable" to reach the answer; that is, we can perhaps believe that we would have gotten there ourselves, if we only managed to persevere long enough. The story goes like this:

$\int \sec x \; dx = \int \frac{dx}{\cos x}$, and $\frac{1}{\cos x}$ is a composition! So try $u = \cos x$, with $du = -\sin x \; dx$.

This means that we write $\int \sec x \; dx = \int \frac{dx}{\cos x} = -\int \frac{-\sin x \; dx}{\sin x \cos x}$.

In order to continue, we need to know how to express $\sin x$ in terms of $u = \cos x$.
But $\sin^2 x = 1 - \cos^2 x$, so $\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-u^2}$. So

$$\int \sec x \; dx = -\int \frac{du}{u\sqrt{1-u^2}}\Big|_{u=\cos x}$$

It is not clear that this is progress, but let's forge on ahead. [At least the trig functions are gone?] In the new integral, maybe we don't like the stuff inside of the square root, so we try $v = 1-u^2$, with $dv = -2u \; du$. [Note that $u^2 = 1-v$.] So

$$-\int \frac{du}{u\sqrt{1-u^2}} = \int \frac{-2u \; du}{2u^2\sqrt{1-u^2}} = \int \frac{dv}{2(1-v)\sqrt{v}}\Big|_{v=1-u^2}$$

Again, it's not quite clear that this is progress, but at least the stuff inside of the square root is less daunting.

But what to do with $\frac{1}{\sqrt{v}}$, or more precisely, $\frac{dv}{2\sqrt{v}}$? Wait, that looks like $d(\sqrt{v})$!

So we try <u>another</u> substitution: $w = \sqrt{v}$, so $dw = \frac{dv}{2\sqrt{v}}$. [Note that $v = w^2$.] So

$$\int \frac{dv}{2(1-v)\sqrt{v}} = \int \frac{1}{1-v}\frac{dv}{2\sqrt{v}} = \int \frac{dw}{1-w^2}\Big|_{w=\sqrt{v}} .$$

Since $1-w^2 = (1-w)(1+w)$, this integral is $\int \frac{dw}{(1-w)(1+w)}$.

Which actually <u>is</u> looking more reasonable. At this point, we reach slightly past what we already know (using "partial fractions") to note that

$$\frac{1}{(1-w)(1+w)} = \frac{1}{2}[\frac{1}{1-w} + \frac{1}{1+w}] + C$$

[which you can verify by putting over a common denominator. Essentially, that is precisely what the partial fractions method is: guessing the right form of the answer and putting it over a common denominator.] But this now yields an integral that we can <u>do</u>!

$$\int \frac{dw}{1-w^2} = \int \frac{1}{2}[\frac{1}{1-w} + \frac{1}{1+w}]\ dw = \frac{1}{2}[-\ln(1-w) + \ln(1+w)] = \frac{1}{2}\ln\big(\frac{1+w}{1-w}\big)$$

by a pair of substitutions. [That makes five integrations by substitution.] Now it is just a matter of rolling back through all of the substitutions we have made! To shorten this, let's do them before we hit the integrals:

$$w = \sqrt{v} = \sqrt{1-u^2} = \sqrt{1-\cos^2 x} = \sqrt{\sin^2 x} = \sin x$$

In other words, we could have gotten to where we found an integral we could solve with one (rather unobvious) substitution! But first, let's just get our answer; we have found that

$$\int \sec x\ dx = \frac{1}{2}\ln\big(\frac{1+w}{1-w}\big)\Big|_{w=\sin x} = \frac{1}{2}\ln\big(\frac{1+\sin x}{1-\sin x}\big) + C$$

Which of course looks nothing like our original answer! But they actually are the same:

$$\begin{aligned}\frac{1}{2}\ln\big(\frac{1+\sin x}{1-\sin x}\big) &= \frac{1}{2}\ln\big(\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\big) = \frac{1}{2}\ln\big(\frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}\big)\\ &= \frac{1}{2}\ln\big(\frac{(1+\sin x)^2}{1-\sin^2 x}\big) = \frac{1}{2}\ln\big(\frac{(1+\sin x)^2}{\cos^2 x}\big) = \frac{1}{2}\ln\big(\frac{(1+\sin x)}{\cos x}\big)^2\\ &= \ln\big(\frac{(1+\sin x)}{\cos x}\big) = \ln\big(\frac{(1}{\cos x} + \frac{\sin x}{\cos x}\big) = \ln(\sec x + \tan x)\end{aligned}$$

What would have happened if we had made the unobvious substitution $u = \sin x$ at the start? $du = \cos x\ dx$ and so

$$\int \sec x\ dx = \int \frac{dx}{\cos x} = \int \frac{\cos x\ dx}{\cos^2 x} = \int \frac{\cos x\ dx}{1-\sin^2 x} = \int \frac{du}{1-u^2}\Big|_{u=\sin x}$$

which is of course where we ended up, with an integral we could do... Although perhaps making that guess for a substitution seems as unlikely as our inventive way of multiplying by one with the original solution! But maybe a lesson to take away from this story, though, is that when you think the right approach is to substitute $u = \cos x$, maybe you should see what happens if to try $u = \sin x$, instead! [And vice versa!]