Math 107H

Topics for the first exam

Integration

Antiderivatives. Integral calculus is all about finding areas of things, e.g. the area between the graph of a function f and the x-axis. This will, in the end, involve finding a function F whose *derivative* is f.

F is an *antiderivative* (or (indefinite) *integral*) of f if $F'(x) = f(x)$.
Notation: $F(x) = \int f(x)\ \mathrm{d}x$; it means $F'(x) = f(x)$; "the integral of f of x dee x"

Basic list:

$\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$) $\qquad \int 1/x\ \mathrm{d}x = \ln|x| + C$

$\int \sin(kx)\ \mathrm{d}x = \dfrac{-\cos(kx}{k} + C \qquad \int \cos(kx)\ \mathrm{d}x = \dfrac{\sin(kx)}{k} + C$

$\int \sec^2 x\ \mathrm{d}x = \tan x + C \qquad \int \csc^2 x\ \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x\ \mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

$\int e^x\ \mathrm{d}x = e^x + C$

$\int \tan x\ \mathrm{d}x = \ln|\sec x| + C \qquad \int \sec x\ \mathrm{d}x = \ln|\sec x + \tan x| + C$

$\int \cot x\ \mathrm{d}x = \ln|\sin x| + C \qquad \int \csc x\ \mathrm{d}x = -\ln|\csc x + \cot x| + C$

Most differentiation rules can be turned into integration rules (although some are harder than others; some we will wait awhile to discover.)
Basic integration rules: sum and constant multiple rules are straighforward to reverse: for k=constant,

$\int k \cdot f(x)\ \mathrm{d}x = k \int f(x)\ \mathrm{d}x \qquad \int (f(x) \pm g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$

Sums and Sigma Notation. Idea: a lot of things can estimated by adding up alot of tiny pieces.

Sigma notation: $\displaystyle\sum_{i=1}^{n} a_i = a_1 + \cdots a_n$; just add the numbers up

Formal properties: $\displaystyle\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

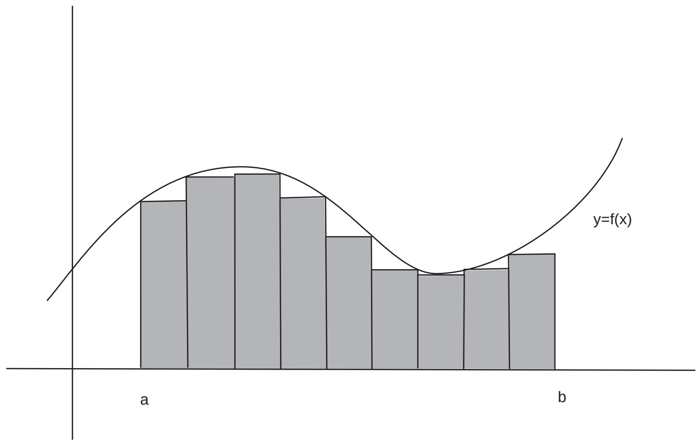
Some things worth adding up:
length of a curve: approximate curve by a collection of straight line segments

length of curve $\approx \sum$(length of line segments)

distance travelled = (average velocity)(time of travel)
over short periods of time, avg. vel. $\approx$ instantaneous vel.
so distance travelled $\approx \sum$(inst. vel.)(short time intervals)

Area and Definite Integrals. Probably the most important thing to approximate by sums: area under a curve.

Idea: approximate region b/w curve and x-axis by things whose areas we can easily calculate: **rectangles**!



Area between graph and x-axis $\approx \sum$ (areas of the rectangles) $=\sum_{i=1}^{n} f(c_i)\Delta x_i$

We <u>define</u> the area to be the <u>limit</u> of these sums as the number of rectangles goes to ∞ (i.e., the width of the rectangles goes to 0), and call this the *definite integral* of f from a to b:

$$\int_a^b f(x)\ dx \ = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

When do such limits exist?

Theorem If f is continuous on the interval $[a,b]$, then $\int_a^b f(x)\ dx$ exists.

(i.e., the area under the graph <u>is</u> approximated by rectangles.)

Properties of definite integrals

First note: the sum used to define a definite integral doesn't <u>need</u> to have $f(x) \geq 0$; the limit still makes sense. When f <u>is</u> bigger than 0, we <u>interpret</u> the integral as area under the graph.

Basic properties of definite integrals:

$$\int_a^a f(x)\ dx = 0 \qquad \int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$$

$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx \qquad \int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$$

$$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$$

If $m \leq f(x) \leq M$ for all x in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$

More generally, if $f(x) \leq g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$

The fundamental theorems of calculus. Formally, $\int_a^b f(x)\ dx$ <u>depends</u> on a and b. Make this explicit:

$$\int_a^x f(t)\ dt = F(x) \text{ is a function of } x.$$

$F(x) =$ the area under the graph of f, from a to x.

Fund. Thm. of Calc (# 2): If f is continuous, then $F'(x) = f(x)$ (F is an antiderivative of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \int_a^b f(t)\ dt$, we get

Fund. Thm. of Calc (# 1): If f is continuous, and F is an antiderivative of f, then $\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\ |_a^b$

Ex: $\int_0^\pi \sin x\ dx = (-\cos\pi) - (-\cos 0) = 2$

Building antiderivatives:

$F(x) = \int_a^x \sqrt{\sin t}\ dt$ is an antiderivative of $f(x) = \sqrt{\sin x}$

$G(x) = \int_{x^2}^{x^3} \sqrt{1+t^2}\ dt = F(x^3) - F(x^2)$, where

$F'(x) = \sqrt{1+x^2}$, so $G'(x) = F'(x^3)(3x^2) - F'(x^2)(2x)$...

Integration by substiution. The idea: reverse the chain rule!

If $g(x) = u$, then $\frac{d}{dx}f(g(x)) = \frac{d}{dx}f(u) = f'(u)\ \frac{du}{dx}$

so $\int f'(u)\ \frac{du}{dx}\ dx = \int f'(u)\ du = f(u) + c$

$\int f(g(x))g'(x)\ dx$; set $u = g(x)$, then $du = g'(x)\ dx$, so $\int f(g(x))g'(x)\ dx = \int f(u)\ du$, <u>where</u> $u = g(x)$

Example: $\int x(x+2-3)^4\ dx$; set $u = x^2 - 3$, so $du = 2x\ dx$. Then

$\int x(x+2-3)^4\ dx = \frac{1}{2}\int (x+2-3)^4 2x\ dx = \frac{1}{2}\int u^4\ du\ |_{u=x^2-3} =$

$\frac{1}{2}\frac{u^5}{5} + c\ |_{u=x^2-3} = \frac{(x^2-3)^5}{10} + c$

The three most important points:

1. Make sure that you calculate (and then set aside) your du before doing step 2!
2. Make sure everything gets changed from x's to u's
3. **<u>Don't</u>** push x's through the integral sign! They're <u>not</u> constants!

We can use u-substitution directly with a definite integral, provided we remember that $\int_a^b f(x)\ dx$ <u>really</u> means $\int_{x=a}^{x=b} f(x)\ dx$, and we remember to change <u>all</u> of the x's to u's!

Ex: $\int_1^2 x(1+x^2)^6\ dx$; set $u = 1 + x^2$, $du = 2x\ dx$. when $x = 1$, $u = 2$; when $x = 2$, $u = 5$; so $\int_1^2 x(1+x^2)^6\ dx = \frac{1}{2}\int_2^5 u^6\ du = ...$

Basic integration formulas (AKA dirty tricks):

change the function without changing the function!

complete the square

$ax^2+bx+c = a(x^2+rx)+c = a(x+r/2)^2+(c-(r/2)^2)$

Ex: $\displaystyle\int \frac{1}{x^2+2x+2}\ \mathrm{d}x = \int \frac{1}{(x+1)^2+1}\ \mathrm{d}x$

use trig identities

$\sin^2 x+\cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sin(2x) = 2\sin x\cos x$, $\dfrac{\tan x}{\sec x}$=$\sin x$, etc.

Ex: $\displaystyle\int \frac{\sin^2 x}{\cos x}\ \mathrm{d}x = \int \frac{1-\cos^2 x}{\cos x}\ \mathrm{d}x = \dots$

pull fractions apart; put fractions together!

Ex: $\displaystyle\int \frac{x+1}{x^3}\ \mathrm{d}x = \int x^{-2}+x^{-3}\ \mathrm{d}x = \dots$

do polynomial long division

Ex: $\displaystyle\int \frac{x^3}{x^2-1}\ \mathrm{d}x = \int x + \frac{x}{x^2-1}\ \mathrm{d}x = \dots$

add zero, multiply by one

Ex: $\int \sec x\ \mathrm{d}x = \int \frac{\sec x(\tan x+\sec x)}{\sec x+\tan x}\ \mathrm{d}x = \dots$ $\qquad \int \frac{x^2}{x^2+1}\ dx = \int \frac{x^2+1-1}{x^2+1}\ dx = \dots$

Integration by parts

Product rule: $\mathrm{d}(uv) = (\mathrm{d}u)v + u(\mathrm{d}v)$

reverse: $\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$

Ex: $\int x\cos x\ \mathrm{d}x$: set u=x, $\mathrm{d}v$=$\cos x\ \mathrm{d}x$ $\qquad$ $\mathrm{d}u$=$\mathrm{d}x$, $v=\sin x$ (or any <u>other</u> antiderivative)

So: $\int x\cos x = x\sin x - \int \sin x\ \mathrm{d}x = \dots$

special case: $\displaystyle\int f(x)\ \mathrm{d}x$; $u = f(x)$, $\mathrm{d}v$=$\mathrm{d}x$ $\qquad \displaystyle\int f(x)\ \mathrm{d}x = xf(x) - \int xf'(x)\ \mathrm{d}x$

Ex: $\displaystyle\int \text{Arcsin}\ x\ \mathrm{d}x = x\ \text{Arcsin}\ x - \int \frac{x}{\sqrt{1-x^2}} = \dots$

The basic idea: integrate part of the function (a part that you <u>can</u>), differentiate the rest. Goal: reach an integral that is “nicer”.

Ex: $\int x^3 \ln x\ dx = (x^4/4)\ln x - \int (x^4/4)(1/x)\ dx = \dots$

Trig substitution

Idea: get rid of square roots, by turning the stuff inside into a perfect square!

$\sqrt{a^2-x^2}$: set $x = a\sin u$. $\mathrm{d}x = a\cos u\ \mathrm{d}u$, $\sqrt{a^2-x^2} = a\cos u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\ \mathrm{d}x = \int \frac{\cos u}{\sin^2 u\cos u}\ \mathrm{d}u\Big|_{x=\sin u} = \dots$

$\sqrt{a^2+x^2}$: set $x = a\tan u$. $\mathrm{d}x = a\sec^2 u\ \mathrm{d}u$, $\sqrt{a^2+x^2} = a\sec u$

Ex: $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\ \mathrm{d}x = \int \frac{2\sec^2 u}{(2\sec u)^3}\ \mathrm{d}u\Big|_{x=2\tan u} = \dots$

$\sqrt{x^2-a^2}$: set $x = a\sec u$. $\mathrm{d}x = a\sec u\tan u\ \mathrm{d}u$, $\sqrt{x^2-a^2} = a\tan u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\ \mathrm{d}x = \int \frac{\sec u\tan u}{\sec^2 u\tan u}\ \mathrm{d}u\Big|_{x=\sec u} = \dots$

Undoing the “u-substitution”: use right triangles! (<u>Draw</u> a right triangle!)

Ex: $x = a\sin u$, then angle u has opposite $= x$, hypotenuse $= a$, so adjacent $= \sqrt{a^2 - x^2}$. So $\cos u = (\sqrt{a^2 - x^2})/a$, $\tan u = x/\sqrt{a^2 - x^2}$, etc.

Trig integrals: What trig substitution usually leads to!

$\int \sin^n x \ \cos^m x \ dx$

If n is odd, keep one $\sin x$ and turn the others, in pairs, into $\cos x$ (using $\sin^2 x = 1 - \cos^2 x$), then do a u-substitution $u = \cos x$.

If m is odd, reverse the roles of $\sin x$ and $\cos x$.

If both are even, turn the $\sin x$ into $\cos x$ (in pairs) and use the double angle formula

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

This will convert $\cos^m x$ into a bunch of *lower powers* of $\cos(2x)$; odd powers can be dealt with by substitution, even powers by another application of the angle doubling formula!

$\int \sec^n x \ \tan^m x \ dx = \int \frac{\sin^m x}{\cos^{n+m} x} \ dx$

If n is *even*, set two of them aside and convert the rest to $\tan x$ using $\sec^2 x = \tan^2 x + 1$, and use $u = \tan x$.

If m is *odd*, set one each of $\sec x$, $\tan x$ aside, convert the rest of the $\tan x$ to $\sec x$ using $\tan^2 x = \sec^2 x - 1$, and use $u = \sec x$.

If n is odd and m is even, convert all of the $\tan x$ to $\sec x$ (in pairs), leaving a bunch of powers of $\sec x$. Then use the *reduction formula*:

$$\int \sec^n x \ dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \ dx$$

At the end, reach $\int sec^2 x \ dx = \tan x + C$ or $\int \sec x \ dx = \ln|\sec x + \tan x| + C$

A little "trick" worth knowing:

the substitution $u = \frac{\pi}{2} - x$, since $\sin(\frac{\pi}{2} - x) = \cos x$ and $\cos(\frac{\pi}{2} - x) = \sin x$, will *reverse* the roles of $\sin x$ and $\cos x$, so will turn $\cot x$ into $\tan u$ and $\csc x$ into $\sec u$. So, for example, the integral

$\int \frac{\cos^4 x}{\sin^7 x} \ dx = \int \csc^3 x \cot^4 x \ dx$, which our techniques don't cover,

becomes $\int \sec^3 u \tan^4 u \ du$, which our techniques <u>do</u> cover.

Partial fractions

rational function = quotient of polynomials

Idea: integrate by writing function as sum of simpler functions

Procedure: $f(x) = p(x)/q(x)$

(0): arrange for degree(p)<degree(q); do long division if it isn't

(1): factor $q(x)$ into linear and irreducible quadratic factors

(2): group common factors together as powers

(3a): for each group $(x-a)^n$ add together: $\dfrac{a_1}{x-a}+\cdots+\dfrac{a_n}{(x-a)^n}$

(3b): for each group $(ax^2+bx+c)^n$ add together:

$$\frac{a_1x+b_1}{ax^2+bx+c}+\cdots+\frac{a_nx+b_n}{(ax^2+bx+c)^n}$$

(4) set $f(x)$ = sum of all sums; solve for the 'undetermined' coefficients

put sum over a common denominator (=$q(x)$); set numerators equal.

always works: multiply out, group common powers, set coeffs of the two polys equal

Ex: $x+3=a(x-1)+b(x-2)=(a+b)x+(-a-2b)$; $1=a+b$, $3=-a-2b$

linear term $(x-a)^n$: set $x=a$, will allow you to solve for a coefficient

if $n\geq 2$, take derivatives of both sides! set x=a, gives another coeff.

$$\text{Ex: } \frac{x^2}{(x-1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}$$
$$=\frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}=\ldots$$

Improper integrals

Fund Thm of Calc: $\displaystyle\int_a^b f(x)\ \mathrm{d}x=F(b)-F(a)$, where $F'(x)=f(x)$

Problems: $a=-\infty$, $b=\infty$; f blows up at a or b or somewhere in between

integral is "improper"; usual technique doesn't work. Solution to this:

$$\int_a^\infty f(x)\ \mathrm{d}x=\lim_{b\to\infty}\int_a^b f(x)\ \mathrm{d}x \qquad \int_{-\infty}^b f(x)\ \mathrm{d}x=\lim_{a\to-\infty}\int_a^b f(x)\ \mathrm{d}x$$

(blow up at a) $\displaystyle\int_a^b f(x)\ \mathrm{d}x=\lim_{r\to a^+}\int_r^b f(x)\ \mathrm{d}x=\lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b f(x)\ \mathrm{d}x$

(similarly for blowup at b (or both!))

$$\int_a^b f(x)\ \mathrm{d}x=\lim_{s\to b^-}\int_a^s f(x)\ \mathrm{d}x=\lim_{\epsilon\to 0^+}\int_a^{b-\epsilon} f(x)\ \mathrm{d}x$$

(blows up at c (b/w a and b)) $\displaystyle\int_a^b f(x)\ \mathrm{d}x=\lim_{r\to c^-}\int_a^r f(x)\ \mathrm{d}x+\lim_{s\to c^+}\int_s^b f(x)\ \mathrm{d}x$

The integral converges if (all of the) limit(s) are finite

Comparison: $0\leq f(x)\leq g(x)$ for all x;

if $\displaystyle\int_a^\infty g(x)\ \mathrm{d}x$ *converges*, so does $\displaystyle\int_a^\infty f(x)\ \mathrm{d}x$

if $\displaystyle\int_a^\infty f(x)\ \mathrm{d}x$ *diverges*, so does $\displaystyle\int_a^\infty g(x)\ \mathrm{d}x$

Numerical Integration (in far too much detail....)

Sometimes (most times?) the Fundamental Theorem of Calculus won't help us to compute a definite integral; we can't find an antiderivative. So we need to fall back on the definition:

$\sum_{i=1}^n f(c_i)\Delta x_i$ approximates $\int_a^b f(x)\ dx$, where the interval $[a,b]$ is cut into n pieces of length $\Delta x_1, \dots \Delta x_n$, and c_i lies in the i-th subinterval

Typically, for convenience, we choose the subintervals to have the same length $\Delta x_i = \Delta x = \frac{b-a}{n}$, and make "standard" choices of elements in the i-th subinterval $[x_{i-1}, x_i]$:

$L(f,n) = \sum_{i=1}^n f(x_{i-1})\Delta x$ (left endpoint estimate)

$R(f,n) = \sum_{i=1}^n f(x_i)\Delta x$ (right endpoint estimate)

$M(f,n) = \sum_{i=1}^n f(\frac{x_{i-1}+x_i}{2})\Delta x$ (midpoint estimate)

Of these, the midpoint estimate is probably best; $L(f,n)$ overestimates area when f is decreasing and underestimates it when f is increasing; $R(f,n)$ does the opposite. $M(f,n)$ tends to average these effects out. In fact, if we know that f'' doesn't get too large, say $|f''(x)| \leq K$ on $[a,b]$, then

$$|\int_a^b f(x)\ dx - M(f,n)| \leq K\frac{(b-a)^3}{24n^2}$$

In the end though, a midpoint estimate is throwing out a lot of information, since it approximates f on an interval by a constant. We can do better, taking into account more infmation about the function f, by approximating f by functions that better "fit" f on a subinterval, whose integrals we know how to compute.

The first is linear functions: we replace f on each subinterval by the linear function having the same values at the endpoints. This essentially replaces a rectangle in our sums with trapezoids. Since the area of a trapezoid is (length of base)(average of lengths of heights), we end up with the estimate

$T(f,n) = \sum_{i=1}^n \frac{f(x_{i-1})+f(x_i)}{2}\Delta x = \frac{1}{2}(\sum_{i=1}^n f(x_{i-1})\Delta x + \sum_{i=1}^n f(x_i)\Delta x)$
$= \frac{1}{2}(L(f,n)+R(f,n))$ (trapezoid estimate)

If f is close to being linear on each subinterval (i.e., f'' is not too big), this gives a better estimate of the integral than either of L or R alone. In fact, if $|f''(x)| \leq K$ on $[a,b]$, then

$$|\int_a^b f(x)\ dx - T(f,n)| \leq K\frac{(b-a)^3}{12n^2}$$

Not quite as good as we expect from midpoints, but it leads us to further impovements. Because: we expect we can do even better if we approximte f by "better" functions, e.g., quadratics!

[Note: our current text takes a somewhat different perspective: whether or not the midpoint and trapezoid rule over- or -under-estimate the integral run <u>opposite</u> to one another: a (weighted) average of the two typically does better than either. Given the estimates of how well they do, we basically average two midpoint and one trapezoid sum together.]

To set this up better, we assume we cut $[a,b]$ into an even number $2n$ of subintervals, so $\Delta x = \frac{b-a}{2n}$. Then we deal with the subintervals in pairs, i.e., with endpoints three at a time:

$$x_{2i}, x_{2i+1} = x_{2i} + \Delta x, x_{2i+2} = x_{2i} + 2\Delta x.$$

There is exactly one quadratic function $g(x) = ax^2+bx+c$ which takes the same value as f at these three points, and by plugging in those values at $x_{2i}, x_{2i}+\Delta x, x_{2i}+2\Delta x$ we get three equations in three unknowns (a, b, and c), which we can solve to determine the quadratic g. This makes a good quadratic approximation to f on the interval $[x_{2i}, x_{2i+2}]$.

But the real point is that we know how to integrate $\int_{x_{2i}}^{x_{2i+2}} g(x)\ dx$ exactly, since it is a quadratic, and a little arithmetic shows that this integral is equal to

$$\int_{x_{2i}}^{x_{2i+2}} g(x)\ dx = \frac{\Delta x}{3}(f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2}))$$

If we sum up these quantities, for each of the n pairs of intervals we have cut $[a, b]$ into, we get *Simpson's Rule*: for $\Delta x = (b - a)/2n$,

$$S(f, n) = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})]$$

It is an amazing fact that this estimate gives the precisely correct integral if f is quadratic <u>or</u> cubic. In fact, how good the estimate is depends on how much the third derivative of f is changing, i.e., on how big the fourth derivative is: if $|f''''(x)| \leq M$ on the interval $[a, b]$, then

$$|\int_a^b f(x)\ dx - S(f, n)| \leq K\frac{(b-a)^5}{180n^4}$$

So, typically, using twice as many intervals (i.e., doing twice the work) gives us an estimate about 16 times closer to the real value of the integral.

The importance of these estimates of the error is that they give us a means to decide beforehand how many subintervals to work with, in order to guarantee that our estimate is within some pre-determined error of the actual value of the integral. Note that, in some sense, every one of these estimates is computed as (length of subinterval)(sum of values of f, one for each subinterval), but some values are weighted more heavily than others. But <u>on</u> <u>average</u> the weight given to a value is one. The trapezoid rule chooses to take half of the values of both endpoints (instead of just one or the other, to avoid playing favorites), and Simpson's Rule gives the middle endpoint of a pair of subintervals twice as much weight as the endpoints.