4

Solutions

Name:

Math 107H Section 1

Final Exam

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

For problems 1 through 3, find the indicated integrals.

1. (10 pts.) $\int_{-1}^{3} \frac{2x}{(2x+3)^{\frac{7}{5}}}\, dx$

$u=(2x+3) \quad 2x=u-3$

$du=2dx \quad dx=\frac{1}{2}du$

$x=-1 \quad u=-2+3=1, \quad x=3 \quad u=6+3=9$

$$= \int_1^9 \frac{u-3}{u^{7/5}}\left(\tfrac{1}{2}du\right)$$

$$= \frac{1}{2}\int_1^9 (u-3)u^{-7/5}\,du = \frac{1}{2}\int_1^9 u^{-2/5}-3u^{-7/5}\,du$$

$$= \frac{1}{2}\left(\frac{5}{3}u^{3/5}-3\left(\frac{-5}{2}\right)u^{-2/5}\right)\Big|_1^9 = \boxed{\frac{1}{2}\left(\left(\frac{5}{3}9^{3/5}+\frac{15}{2}9^{-2/5}\right)-\left(\frac{5}{3}\cdot 1+\frac{15}{2}(1)\right)\right)}$$

$$= \frac{1}{2}\left(\left(\frac{5}{3}\cdot 9+\frac{15}{2}\right)9^{-2/5}-\left(\frac{10+45}{6}\right)\right) = \frac{45}{4}9^{-2/5}-\frac{55}{12}$$

$\longrightarrow$ alternate solution at end

2. (10 pts.) $\int x\sin(4x)\, dx$

$u=x \quad du=dx$

$dv=\sin(4x)dx \quad v=-\frac{1}{4}\cos(4x)$

$$= -\frac{1}{4}x\cos(4x)+\frac{1}{4}\int \cos(4x)\,dx$$

$$= -\frac{1}{4}x\cos(4x)+\frac{1}{4}\left(\frac{1}{4}\sin(4x)\right)+C.$$

$$= -\frac{1}{4}x\cos(4x)+\frac{1}{16}\sin(4x)+C$$

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3. (15 pts.) $\displaystyle\int \frac{x^2}{(x+1)(x+2)(x+3)}\,dx$

$$= \int \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}\,dx = (*)$$

$$\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} = \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{(x+1)(x+2)(x+3)}$$

$$x^2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

$x=-1$ $\quad (-1)^2=1 = A(1)(2) + B(0) + C(0) = 2A \quad A=\frac{1}{2}$

$x=-2$ $\quad (-2)^2=4 = A(0) + B(-1)(1) + C(0) = -B \quad B=-4$

$x=-3$ $\quad (-3)^2=9 = A(0) + B(0) + C(-2)(-1) = 2C \quad C=9/2$

So $\displaystyle (*) = \int \frac{1}{2}\frac{1}{x+1} - 4\frac{1}{x+2} + \frac{9}{2}\frac{1}{x+3}\,dx$

$$= \frac{1}{2}\ln|x+1| - 4\ln|x+2| + \frac{9}{2}\ln|x+3| + C$$

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4. For the integrals below, when the appropriate substitution is made, what (trigonometric) integral results? **Express your integrand in terms of** $\sin x$ **and** $\cos x$.

(a) (10 pts.) $\int \frac{\sqrt{x^2-2}}{x^2}\,dx$

$x=\sqrt{2}\sec u \quad dx=\sqrt{2}\sec u\tan u\,du$

$x^2-2=2\sec^2 u-2=2\tan^2 u$

$$=\int \frac{\sqrt{2}\tan u}{(\sqrt{2}\sec u)^2}\sqrt{2}\sec u\tan u\,du\Big|_{x=\sqrt{2}\sec u} = \int \frac{\tan^2 u}{\sec u}\,du\Big|_{x=\sqrt{2}\sec u}$$

$$=\int \frac{\frac{\sin^2 u}{\cos^2 u}}{\frac{1}{\cos u}}\,du\Big|_{x=\sqrt{2}\sec u} = \int \frac{\sin^2 u}{\cos u}\Big|_{x=\sqrt{2}\sec u}.$$

(b) (10 pts.) $\int \frac{x^2}{\sqrt{3-x^2}}\,dx$

$x=\sqrt{3}\sin u \quad dx=\sqrt{3}\cos u\,du$

$3-x^2=3-3\sin^2 u=3\cos^2 u$

$$=\int \frac{(\sqrt{3}\sin u)^2}{\cancel{\sqrt{3}}\cancel{\cos u}}\cancel{\sqrt{3}}\cancel{\cos u}\,du\Big|_{x=\sqrt{3}\sin u} = \int 3\sin^2 u\,du\Big|_{x=\sqrt{3}\sin u}$$

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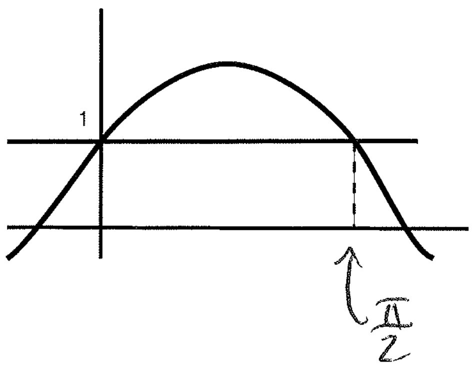
5. (15 pts.) Find the volume of the region R obtained by revolving the region A lying below the graph of $f(x) = \sin x + \cos x$ and above the line $y = 1$, from $x = 0$ to the next time the graph meets the line, around the x-axis. (See figure.)



$\sin x + \cos x = 1$

$(\sin x + \cos x)^2 = 1^2 = 1 = \sin^2 x + 2\sin x \cos x + \cos^2 x$

$0 = 2\sin x \cos x$

$\sin x = 0$ or $\cos x = 0$

$x = 0, \pi, 2\pi, .$ $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$$\text{Volume} = \int_0^{\frac{\pi}{2}} \pi (R^2 - r^2)\, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 - 1^2\, dx = \pi \int_0^{\frac{\pi}{2}} \sin^2 x + 2\sin x \cos x + \cos^2 x - 1\, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} 1 + 2\sin x \cos x - 1\, dx = \pi \int_0^{\pi/2} 2\sin x \cos x\, dx$$

$u = \sin x \quad du = \cos x\, dx$

$x = 0 \to u = 0, \ x = \frac{\pi}{2} \to u = 1$

$$= \pi \int_0^1 2u\, du = \pi u^2 \Big|_0^1 = \pi(1-0)$$

$$= \underline{\underline{\pi}}$$

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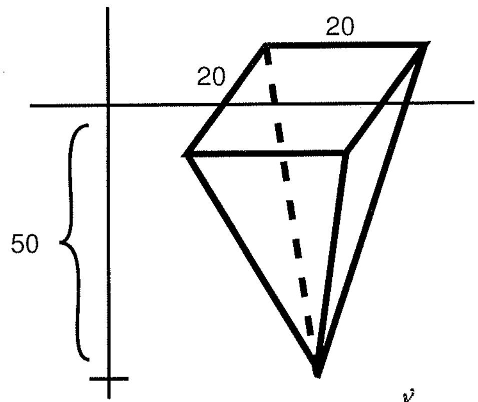
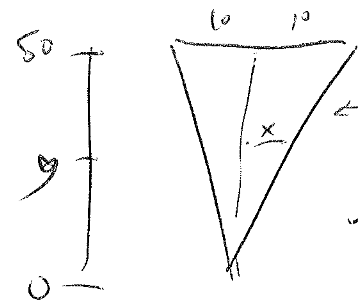
6. (15 pts.) Find the work done in digging out a hole shaped like an inverted square pyramid, whose point is at a depth of 50 feet, and whose cross sections grow to be a square that has sides of length 20 feet at ground level. Compacted earth, like that from your hole, has a weight of 100 pounds per cubic foot. (See figure!)





$x = \frac{1}{5}y$

$l = \frac{2}{5}y$

$A = \left(\frac{2y}{5}\right)^2$

$$\text{Work} = \int_{\text{bottom}}^{\text{top}} \overset{\text{density}}{}(\text{distance})(\text{area})\,dy$$

$$= \int_0^{50} 100\,(50-y)\left(\frac{2y}{5}\right)^2 dy = \int_0^{50} 16\,y^2(50-y)\,dy$$

$$= 16\int_0^{50} 50y^2 - y^3\,dy = 16\left(\frac{50}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^{50}$$

$$= 16\left(\frac{50}{3}(50)^3 - \frac{1}{4}(50)^4\right) = 16(50)^4\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 16(50)^4\left(\frac{1}{12}\right) = \frac{4}{3}(50)^4.$$

$\longrightarrow$ alternate solution at end.

7. (10 pts. each) Determine whether or not each of the following series converges.

(a): $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+n+11}}$ limit compare to $\sum \frac{n}{\sqrt{n^3}} = \sum \frac{n}{n^{3/2}} = \sum \frac{1}{n^{1/2}}$

$'' \sum a_n$

$$\frac{n}{\sqrt{n^3+n+11}} \Big/ \frac{n}{\sqrt{n^3}} = \sqrt{\frac{n^3}{n^3+n+11}} = \sqrt{\frac{1}{1+\frac{1}{n^2}+\frac{11}{n^3}}} \longrightarrow \sqrt{\frac{1}{1+0+0}} = 1 \neq 0, \infty$$

$\underline{\text{so}}$ since $\sum 1/n^{1/2}$ diverges (p-series, $p = 1/2 < 1$),

$\sum a_n$ diverges $\longrightarrow$ alternate solution at end of solutions.

(b): $\sum_{n=0}^{\infty} ne^{-n^2} = \sum a_n$ $\quad a_n = f(n)$ $\quad f(x) = xe^{-x^2}$

$$f'(x) = e^{-x^2} + x(-2xe^{-x^2}) = e^{-x^2}(1-2x^2) < 0$$

for $x \geq 1$, so

f is decreasing.

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{N\to\infty} \int_0^N xe^{-x^2}\,dx \qquad \begin{aligned} u &= -x^2 \\ du &= -2x\,dx \\ x\,dx &= -\tfrac{1}{2}du \end{aligned}$$

$$= \lim_{N\to\infty} \int_0^{-N^2} -\tfrac{1}{2}e^u\,du = \lim -\tfrac{1}{2}e^u \Big|_0^{-N^2} = \lim_{N\to\infty} -\tfrac{1}{2}e^{-N^2} + \tfrac{1}{2}$$

$= 0 + \frac{1}{2} = \frac{1}{2} < \infty$, so $\sum a_n$ converges by the integral test.

$\longrightarrow$ alternate solution at end of solutions.

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8. (15 pts.) Find the **interval** of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n}{2^n + 3^n}(x-1)^n = \sum a_n (x-1)^n$$

$$a_n = \frac{n}{2^n+3^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}+3^{n+1}}}{\frac{n}{2^n+3^n}} = \left(\frac{n+1}{n}\right)\left(\frac{2^n+3^n}{2^{n+1}+3^{n+1}}\right)$$

$$= \left(\frac{n+1}{n}\right)\left(\frac{(2/3)^n+1}{2(2/3)^n+3}\right) \to (1)\left(\frac{0+1}{2(0)+3}\right) = \frac{1}{3} = L$$

so $R = \frac{1}{L} = 3$ = radius of convergence

$x-1=-3$
$x=-2$

$x-1=3$
$x=4$

$$\sum \frac{n}{2^n+3^n}(-3)^n =$$
$$= \sum (-1)^n n \frac{3^n}{2^n+3^n} \to 1$$
$$= \sum (-1)^n a_n$$

but $a_n \to \infty$ as $n \to \infty$,

so series diverges by nth term test

$$\sum \frac{n}{2^n+3^n}(3)^n$$
$$= \sum n \frac{3^n}{2^n+3^n} \to 1$$

terms $\to \infty$ as $n \to \infty$

so series diverges

interval of convergence = $(-2, 4)$

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9. (15 pts.) Find the Taylor polynomial of degree 3, $P_3(x)$, centered at $a = 0$, for the function $f(x) = (8 + x)^{2/3}$.

$f(x) = (x+8)^{2/3}$

$f'(x) = \frac{2}{3}(x+8)^{-1/3}$

$f''(x) = \frac{-2}{9}(x+8)^{-4/3}$

$f'''(x) = \left(-\frac{2}{9}\right)\left(\frac{-4}{3}\right)(x+8)^{-7/3} = \frac{8}{27}(x+8)^{-7/3}$

So:

$f(0) = 8^{2/3} = 2^2 = 4$

$f'(0) = \frac{2}{3} 8^{-1/3} = \frac{2}{3} \frac{1}{2} = \frac{1}{3}$

$f''(0) = -\frac{2}{9} 8^{-4/3} = \frac{-2}{9} \cancel{2}^{-4} = \frac{-1}{9 \cdot 8} = \frac{-1}{72}$

$f'''(0) = \frac{8}{27} 2^{-7} = \frac{1}{27} \cdot \frac{1}{16}$

So:

$\cancel{P} P_3(x) = 4 + \frac{1}{3}(x-0) - \frac{1}{72} \cdot \frac{1}{2} x^2 + \frac{1}{27} \frac{1}{16} \cdot \frac{1}{6} x^3$

$= 4 + \frac{1}{3}x - \frac{1}{144}x^2 + \frac{1}{2592}x^3$

$$\begin{array}{r} 96 \\ 27 \\ \hline 672 \\ 192 \\ \hline 2592 \end{array}$$

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10. (15 pts.) Starting from the Taylor series for $f(x) = \dfrac{1}{1-x}$ centered at $a = 0$, show how to build (by multiplication, substitution, differentiation, and/or integration) the Taylor series for the function

$$g(x) = \frac{\ln(1+x^3)}{x}$$

(also) centered at $a = 0$.

[Hint: start by building $h(x) = \dfrac{1}{1+x}$ (!).]

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

So:

$$\ln(1+x) = \int \frac{dx}{1+x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{(integration term-by-term)}$$

So:

$$\ln(1+x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{n+1}$$

So!!

$$\frac{\ln(1+x^3)}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{3n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{3n+2} .$$

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or: $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$; $\dfrac{1}{1+x^3} = \displaystyle\sum_{n=0}^{\infty} (-1)^n (x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$

$$\frac{3x^2}{1+x^3} = (3x^2) \sum (-1)^n x^{3n} = 3 \sum_{n=0}^{\infty} (-1)^n x^{3n+2} \text{; integrate!}$$

$$\ln(1+x^3) = 3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{3n+3} \text{ ; } \frac{\ln(1+x^3)}{x} = \frac{3}{x} \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{3n+3}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{n+1}$$

1. By parts!

$u = 2x \quad dv = (2x+3)^{-7/5}\,dx$

$du = 2\,dx \quad v = \frac{1}{2}(2x+3)^{-2/5}\left(\frac{-5}{2}\right)$

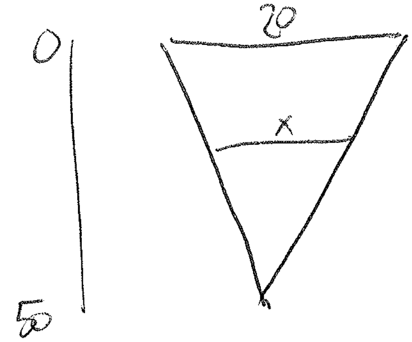
$$\int_{-1}^{3} \frac{2x}{(2x+3)^{7/5}}\,dx = \int_{-1}^{3} 2x(2x+3)^{-7/5}\,dx$$

$$= \frac{-5}{2}x(2x+3)^{-2/5}\Big|_{-1}^{3} - \int_{-1}^{3}\left(\frac{-5}{2}\right)(2x+3)^{-2/5}\,dx$$

$$= -\frac{5}{2}x(2x+3)^{-2/5} + \frac{5}{2}\left(\frac{1}{2}(2x+3)^{3/5}\right)\left(\frac{5}{3}\right)\Big|_{-1}^{3}$$

$$= -\frac{5}{2}\left(3(9)^{-2/5} - (-1)(1)^{-2/5}\right) + \frac{25}{12}\left(9^{3/5} - 1^{3/5}\right)$$

6.



$y=0 \quad x=20$

$y=50 \quad x=0$

$\longrightarrow \quad x = 20 - \frac{2}{5}y$

$$\text{Work} = \int_0^{50} (\text{density})(\text{area})(\text{distance})\,dy$$

$$= \int_0^{80} (100)\left(20-\frac{2}{5}y\right)^2 (y)\,dy$$

$$= \int_0^{80} 100\left(400 - \frac{80}{5}y + \frac{4}{25}y^2\right) y\,dy$$

$$= 100\int_0^{50} 400y - 16y^2 + \frac{4}{25}y^3\,dy$$

$$= 100\left(200y^2 - \frac{16}{3}y^3 + \frac{1}{25}y^4\right)\Big|_0^{50}$$

$$= 100\left(200(50)^2 - \frac{16}{3}(50)^3 + \frac{1}{25}(50)^4\right)$$

Alternate solutions:

7(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+n+11}} = (*)$

$n^3 \geq n$ and $n^3 \geq 11$ for $\underline{n \geq 3}$.

$$\frac{n}{\sqrt{n^3+n+11}} \geq \frac{n}{\sqrt{n^3+n^3+n^3}} \quad \text{since}$$

$$= \frac{n}{\sqrt{3n^3}} = \frac{n}{\sqrt{3}\, n^{3/2}} = \frac{1}{\sqrt{3}} \frac{1}{n^{1/2}}$$

Then since $\sum \frac{1}{n^{1/2}}$ diverges (p-series, $p = 1/2 \leq 1$),

$\sum \frac{1}{\sqrt{3}\, n^{1/2}}$ diverges, so $(*)$ diverges by comparison.

7(b) $\displaystyle\sum_{n=0}^{\infty} n e^{-n^2} = \sum a_n = (*)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1) e^{-(n+1)^2}}{n\, e^{-n^2}} = \left(\frac{n+1}{n}\right) e^{-(n+1)^2+n^2} = \left(\frac{n+1}{n}\right) e^{-2n-1}$$

$$\to (1)(0) \text{ as } n \to \infty, \text{ so}$$

$(*)$ converges, by the Ratio Test.

[comparison, and limit comparison, also work!].