Math 107H, Section 1

Quiz number 3 Solutions

4. For the following integrals, if you make the substitution $u = \ln x$, what does the integral become?

(a) $\displaystyle\int \frac{dx}{x\ln x}$

Setting $u = \ln x$, we have $du = \dfrac{1}{x}\,dx$, and so

$$\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}\frac{1}{x}\,dx = \int \frac{1}{u}\,du\Big|_{u=\ln x}$$

(b) $\displaystyle\int \frac{dx}{\ln x}$

Setting $u = \ln x$, we have $du = \dfrac{1}{x}\,dx$ and (since we will need it) $x = e^{\ln x} = e^u$, and so

$$\int \frac{dx}{\ln x} = \int \frac{1}{\ln x}x(\frac{1}{x}\,dx) = \int \frac{1}{u}e^u\,du\Big|_{u=\ln x} = \int \frac{e^u}{u}\,du\Big|_{u=\ln x}$$

One of these integration problems you can finish! Finish it. (The other one is known as the *logarithmic integral* $\mathrm{Li}(x)$, and we <u>cannot</u> finish it; it is a(nother) "special" function.)

We can finish the first one!

$$\int \frac{dx}{x\ln x} = \int \frac{1}{u}\,du\Big|_{u=\ln x} = \ln|u| + C\Big|_{u=\ln x} = \ln|\ln x| + C$$

N.B. $\mathrm{Li}(x) = \displaystyle\int_2^x \frac{dt}{\ln t}$ cannot be expressed as a combination of our "elementary" functions.

Note that we have therefore shown that $\mathrm{Ei}(x) = \displaystyle\int_2^x \frac{e^t}{t}\,dt$ (known as the *exponential integral function*) <u>also</u> cannot be expressed in elementary terms; if it could, our u-substitution would let us do the same for $\mathrm{li}(x)$!