Solutions

Name:

Math 107H Exam 1

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Find each of the following integrals.

Note that " $\int_3^x f(t)\ dt + C$ " is not a sufficient computation of an antiderivative!

Some formulas of potential use can be found at the bottom of the last page of the exam.

1. (10 pts.) $\int (x+2)^{3/2}\ dx$

$u = x+2 \qquad du = dx$

$$\int (x+2)^{3/2} dx = \int u^{3/2} du \Big|_{u=x+2} = \frac{2}{5} u^{\frac{5}{2}} + C \Big|_{u=x+2}$$

$$\boxed{= \frac{2}{5}(x+2)^{5/2} + C}$$

3 is odd!

2. (15 pts.) $\int_0^{\pi/2} \sin^3 x \, dx$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \, (\sin x \, dx)$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)(\sin x \, dx)$$

$u = \cos x \quad du = -\sin x \, dx$

$x = 0 \leadsto u = 1$

$x = \frac{\pi}{2} \leadsto u = 0$

$$= \int_1^0 (1 - u^2)(-du) = \int_1^0 u^2 - 1 \, du$$

$$= \frac{u^3}{3} - u \Big|_1^0 = \boxed{(0 - 0) - \left(\frac{1}{3} - 1\right)}$$

$$= 0 - \left(\frac{-2}{3}\right) = \frac{2}{3}$$

3. (10 pts.) $\int \frac{x^2+x-3}{x^{1/2}}\,dx$ $= \int \frac{x^2}{x^{1/2}} + \frac{x}{x^{1/2}} - \frac{3}{x^{1/2}}\,dx$

$= \int x^{3/2} + x^{1/2} - 3x^{-1/2}\,dx$

$\boxed{= \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} - 3(2x^{1/2}) + C}$

4. (15 pts.) $\int_0^1 e^{\sqrt{x}}\, dx$

$u = \sqrt{x}$ ${}_{''}x^{\frac{1}{2}}$ $du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}dx$

$u=0 \to x=0$

$u=1 \sim x=1$

$$= \int_0^1 (2\sqrt{x})e^{\sqrt{x}}\left(\frac{dx}{2\sqrt{x}}\right) = \int_0^1 2ue^u\,du$$

$v = u \quad dw = e^u du$

$dv = du \quad w = e^u$

$$= 2\int_0^1 ue^u\,du$$

$$= 2\left(ue^u\Big|_0^1 - \int_0^1 e^u du\right)$$

$$= 2\left(ue^u\Big|_0^1 - e^u\Big|_0^1\right)$$

$$\boxed{= 2\left((1\cdot e - 0\cdot 1) - (e-1)\right)}$$

$$= 2(e - 0 - e + 1) = 2(1) = \underline{\underline{2}}.$$

$u = e^{\sqrt{x}}$ also works! Gets you to $2\int \ln u\,du\Big|_{u=e^{\sqrt{x}}}\ldots$

5. (15 pts.) $\displaystyle\int \frac{dx}{(x+1)^2(x+4)} = (*)$

$$\frac{1}{(x+1)^2(x+4)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+4)}$$

$$= \frac{A(x+1)(x+4) + B(x+4) + C(x+1)^2}{(x+1)^2(x+4)}$$

So need: $1 = A(x+1)(x+4) + B(x+4) + C(x+1)^2$

Set $x=-1$: $1 = A(0)(3) + B(3) + C(0)^2 = 3B$

$\Rightarrow B = 1/3$

Set $x=-4$: $1 = A(-3)(0) + B(0) + C(-3)^2 = 9C$

$\Rightarrow C = 1/9$

Set $x=0$: $1 = A(1)(4) + \frac{1}{3}(4) + \frac{1}{9}(1)^2 = 4A + \frac{4}{3} + \frac{1}{9}$

$\Rightarrow 4A = 1 - \frac{4}{3} - \frac{1}{9} = \frac{9-12-1}{9} = \frac{-4}{9} \Rightarrow A = -\frac{1}{9}$

So $(*) = \displaystyle\int -\frac{1}{9}\frac{1}{x+1} + \frac{1}{3}\frac{1}{(x+1)^2} + \frac{1}{9}\frac{1}{x+4}\,dx$

($u=x+1$, $du=dx$) ($u=x+1$, $du=dx$) ($u=x+4$, $du=dx$)

$$\boxed{= -\frac{1}{9}\ln|x+1| + \frac{1}{3}\left(\frac{-1}{(x+1)}\right) + \frac{1}{9}\ln|x+4| + C}$$

6. (15 pts.) $\boxed{\int e^{-x} \sin(3x)\, dx}$

(!)

$u = \sin(3x) \quad dv = e^{-x}dx$

$du = 3\cos(3x)dx \quad v = -e^{-x}$

$= -e^{-x}\sin 3x - \int -3e^{-x}\cos(3x)\,dx$

$= -e^{-x}\sin 3x + 3\int e^{-x}\cos(3x)\,dx$

$u = \cos(3x) \quad dv = e^{-x}dx$

$du = -3\sin(3x) \quad v = -e^{-x}$

$= -e^{-x}\sin 3x + 3\left(-e^{-x}\cos(3x) - \int -(-3)e^{-x}\sin 3x\,dx\right)$

$= -e^{-x}\sin 3x - 3e^{-x}\cos 3x - 9\boxed{\int e^{-x}\sin 3x\,dx}$

So $10\int e^{-x}\sin 3x\,dx = -e^{-x}\sin 3x - 3e^{-x}\cos 3x$

So: $\boxed{\int e^{-x}\sin 3x\,dx = -\frac{1}{10}e^{-x}\sin 3x - \frac{3}{10}e^{-x}\cos 3x + C}$

7. (20 pts.) $\int (x^2+1)^{3/2}\,dx = \int \left(\sqrt{x^2+1}\,\right)^3 dx$

Think: $\tan^2 u + 1 = \sec^2 u$

$x = \tan u \quad dx = \sec^2 u\, du$

$\cancel{dx} \quad x^2+1 = \tan^2 u + 1 = \sec^2 u$

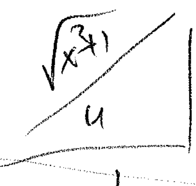
$\sqrt{x^2+1} = \sec u$

$$= \int (\sec u)^3 (\sec^2 u\, du)\Big|_{x=\tan u}$$

$$= \int \sec^5 u\, du\Big|_{x=\tan u} \overset{(n=5)}{=} \frac{1}{4}\sec^3 u \tan u\Big|_{x=\tan u} + \frac{3}{4}\int \sec^3 u\, du\Big|_{x=\tan u}$$

$$\overset{(n=3)}{=} \frac{1}{4}\sec^3 u\tan u + \frac{3}{4}\left(\frac{1}{2}\sec u \tan u + \frac{1}{2}\int \sec u\, du\right)\Big|_{x=\tan u}$$

$$= \frac{1}{4}\sec^3 u \tan u + \frac{3}{8}\sec u \tan u + \frac{3}{8}\ln|\sec u + \tan u| + C\Big|_{x=\tan u}$$



$\longrightarrow \tan u = x$

$\sec u = \sqrt{x^2+1}$

$$= \frac{1}{4}\left(\sqrt{x^2+1}\right)^3 x + \frac{3}{8}\left(\sqrt{x^2+1}\right)x + \frac{3}{8}\ln\left|\sqrt{x^2+1} + x\right| + C$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

$$c^2 \int \frac{dy}{(y^2+c^2)^k} = \frac{1}{(2k-2)}\cdot\frac{y}{(y^2+c^2)^{k-1}} + \frac{(2k-3)}{(2k-2)}\int \frac{dy}{(y^2+c^2)^{k-1}}$$

1. Find the following integrals (10 pts. each):

(a): $\int_1^4 x^2 \ln x \, dx$ by parts! $u = \ln x$ $dv = x^2 dx$
$du = \frac{1}{x}dx$ $v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \ln|x| \Big|_1^4 - \int_1^4 \frac{1}{3}x^2 dx = \frac{1}{3}x^3 \ln|x| \Big|_1^4 - \frac{1}{9}x^3 \Big|_1^4$$

$$= \left(\frac{1}{3}4^3 \ln(4) - \frac{1}{3}1^3 \ln(1)\right) - \frac{1}{9}(4^3 - 1^3)$$

[can also be done by $u = x^2 \ln x$, $dv = dx$!]

(b): $\int \sin^2 x \cos^3 x \, dx$ $= \int \sin^2 x \cos^2 x \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$ $u = \sin x$ $du = \cos x \, dx$

$$= \int u^2 (1 - u^2) \, du \Big|_{u = \sin x} = \int u^2 - u^4 du \Big|_{u = \sin x}$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c \Big|_{u = \sin x} = \frac{1}{3}(\sin x)^3 - \frac{1}{5}(\sin x)^5 + C \quad //$$

2. When you apply the appropriate trigonometric substitutions, what do the following integrals become?

(a): $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

$x = 2\sin u \quad dx = 2\cos u\,du$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2 u} = 2\cos u$

$$= \boxed{\int \frac{(2\cos u)(2\cos u\,du)}{(2\sin u)^2}}\Bigg|_{x=2\sin u} = \int \frac{\cos^2 u}{\sin^2 u}\,du\Bigg|_{x=2\sin u}$$

(b): $\int \frac{x^2}{\sqrt{4x^2+9}}\,dx$

want $\quad 4x^2+9 = 9\tan^2 u + 9 = 9\sec^2 u$

$2x = 3\tan u \quad x = \frac{3}{2}\tan u$

$dx = \frac{3}{2}\sec^2 u\,du$

$\sqrt{4x^2+9} = \sqrt{9\sec^2 u} = 3\sec u$

$$= \boxed{\int \frac{(\frac{3}{2}\tan u)^2(\frac{3}{2}\sec^2 u\,du)}{3\sec u}}\Bigg|_{x=\frac{3}{2}\tan u} = \int \frac{3}{4}\tan^2 u\sec u\,du\Bigg|_{x=\frac{3}{2}\tan u}$$

6. (15 pts.) Recall that if a function f has second derivative satisfying $|f''(x)| \le M$ for every x in the interval $[a, b]$, then the error E_n in approximating the integral $\int_a^b f(x)\,dx$ using the trapezoidal rule using n equal subintervals is at most

$$M\frac{(b-a)^3}{12n^2}$$

Based on this, how many subintervals should we divide the interval $[2, 5]$ into in order to be sure to approximate the integral $\int_2^5 x \ln x \, dx$ with an error of less than $\frac{1}{100}$?

$f(x) = x\ln x$

$f'(x) = 1\cdot\ln x + x\left(\frac{1}{x}\right) = \ln x + 1$

$f''(x) = \frac{1}{x} + 0 = \frac{1}{x}$ ← decreasing! $\left(f'''(x) = -\frac{1}{x^2} < 0\right)$

so max is at $x = 2$ $\quad \underline{M = \frac{1}{2}}$

want error $< \frac{1}{100}$, error $\le \frac{1}{2}\frac{(5-2)^3}{12n^2}$

so want $\frac{1}{2}\frac{3^3}{12n^2} < \frac{1}{100}$ so $n^2 > 100 \cdot \frac{1}{2} \cdot \frac{3^3}{12}$

$= 50 \cdot \frac{27}{12} = 50 \cdot \frac{9}{4}$

$= \frac{450}{4} = 112.5$

so need $n > \sqrt{112.5} \approx 10.6$

so $n = 11$ will work.

Math 107+1 Exam 1 Solutions

1. $\displaystyle\int_2^3 (3x-2)^{5/4}\,dx$ $\qquad u=3x-2,\ du=3dx,\ dx=\frac{1}{3}du$

$\qquad x=2 \to u=6-2=4,\ x=3 \to u=9-2=7$

$$=\int_4^7 u^{5/4}\left(\tfrac{1}{3}\right)du = \tfrac{1}{3}\left(\tfrac{4}{9}u^{\frac{9}{4}}\right)\Big|_4^7 = \tfrac{4}{27}\left(7^{9/4}-4^{9/4}\right).$$

[or: solve indefinite integral first, then plug in values.]

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2. $\displaystyle\int x^{\frac{1}{3}}\ln x\,dx$ $\qquad u=\ln x \quad dv=x^{\frac{1}{3}}dx$

$\qquad du=\frac{1}{x}dx \quad v=\frac{3}{4}x^{4/3}$

$$=uv-\int v\,du = \tfrac{3}{4}x^{4/3}\ln x-\int \tfrac{3}{4}x^{4/3}\tfrac{1}{x}dx$$

$$=\tfrac{3}{4}x^{4/3}\ln x-\tfrac{3}{4}\int x^{1/3}dx = \tfrac{3}{4}x^{4/3}\ln x-\tfrac{3}{4}\left(\tfrac{3}{4}x^{4/3}\right)+C$$

$$=\tfrac{3}{4}x^{4/3}\ln x-\tfrac{9}{16}x^{4/3}+C.$$

[or: u-subs! $u=\ln x,\ du=\frac{1}{x}dx,\ dx=x\,du=e^u du$

$x^{1/3}=(e^u)^{\frac{1}{3}}=e^{u/3}$; gives $\displaystyle\int u e^{\frac{4}{3}u}du\Big|_{u=\ln x}$]

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3. $$\int_0^{\frac{\pi}{2}}\cos^3x\,dx = \int_0^{\frac{\pi}{2}}(1-\sin^2x)(\cos x\,dx) = \int_0^{\frac{\pi}{2}}(1-\sin^2x)\cos x\,dx$$

$u=\sin x \quad du=\cos x\,dx \quad x=0\to u=0 \quad x=\frac{\pi}{2}\to u=1$

$$=\int_0^1(1-u^2)\,du = u-\frac{u^3}{3}\Big|_0^1 = \left(1-\tfrac{1}{3}\right)-(0-0)=\tfrac{2}{3}$$

4. $\displaystyle\int \frac{dx}{1+\sqrt{x}}$ $\quad u=\sqrt{x}=x^{\frac{1}{2}}$ $\quad du=\frac{1}{2}x^{-\frac{1}{2}}dx=\frac{1}{2\sqrt{x}}dx$

$$=\int \frac{1}{1+\sqrt{x}}(2\sqrt{x})\left(\frac{1}{2\sqrt{x}}dx\right)=\int \frac{2\sqrt{x}}{1+\sqrt{x}}\left(\frac{1}{2\sqrt{x}}dx\right)=\int\frac{2u}{1+u}du\Big|_{u=\sqrt{x}}$$

$$=\int \frac{2(u+1)-2}{1+u}du\Big|_{u=\sqrt{x}}=\int 2-\frac{2}{u+1}du\Big|_{u=\sqrt{x}}=2u-2\ln|u+1|+C\Big|_{u=\sqrt{x}}$$

$$=2\sqrt{x}-2\ln|\sqrt{x}+1|+C.$$

[or: $u=1+\sqrt{x}$ works: $=\displaystyle\int \frac{2(u-1)}{u}du\Big|_{u=1+\sqrt{x}}=\dots$]

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5. $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$: Set $x=\sin u$ $\quad dx=\cos u\,du$, so
$\sqrt{1-x^2}=\cos u$

$$=\int\frac{\cos u\,du}{(\sin^2u)\cos u}\Big|_{x=\sin u}=\int\frac{du}{\sin^2u}\Big|_{x=\sin u}=\int\csc^2u\,du\Big|_{x=\sin u}$$

$\displaystyle\int\frac{x^2}{\sqrt{x^2+9}}dx$ Set $x=3\tan u$ $\quad dx=3\sec^2u\,du$, so
$\sqrt{x^2+9}=3\sec u$

$$=\int\frac{(3\tan u)^2}{3\sec u}(3\sec^2u\,du)\Big|_{x=3\tan u}=9\int\tan^2u\sec u\,du\Big|_{x=3\tan u}$$

6. $\frac{1}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5} = \frac{A(x+5)+B(x+2)}{(x+2)(x+5)}$

so $1 = A(x+5)+B(x+2)$ $\quad x=-5: \ 1 = 0 + B(-3) \quad B = -1/3$

$x=-2 \quad 1 = A(3)+0 \quad A = 1/3$

so $\frac{1}{(x+2)(x+5)} = \frac{1}{3}\frac{1}{x+2} - \frac{1}{3}\frac{1}{x+5}$

For $\frac{1}{(x^2+2)(x^2+5)}$, replace x with x^2! $\boxed{\frac{1}{3}\frac{1}{x^2+2} - \frac{1}{3}\frac{1}{x^2+5}}$.

So $\int \frac{dx}{(x^2+2)(x^2+5)} = \int \frac{1}{3}\frac{1}{x^2+2} - \frac{1}{3}\frac{1}{x^2+5}\,dx = \frac{1}{3}\int\frac{dx}{x^2+(\sqrt{2})^2} - \frac{1}{3}\int\frac{dx}{x^2+(\sqrt{5})^2}$

$= \frac{1}{3}\frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{3}\frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)$.

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7. Error $\le M\frac{(b-a)^3}{24n^2} < \frac{1}{100}$ $\quad b-a = 5-1 = 4$

$f(x) = \frac{1}{x}$, so $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$.

$f''(x)$ is decreasing on $[1,5]$ (its derivative is $\frac{-6}{x^4} < 0$)

so max of $f''(x)$ as at $x=1$: $M = 2$. So we want

$2\frac{(4)^3}{24n^2} < \frac{1}{100}$, so $\frac{2(4^3)100}{24} < n^2$ ~~$= \frac{128}{24}\cdot 100$~~

so $n^2 > \frac{2\cdot 64}{24}\cdot 100 = \frac{64}{12}\cdot 100 = \frac{16}{3}\cdot 100$ so $n > \sqrt{\frac{16}{3}\cdot 100}$

$= \frac{\sqrt{16}}{\sqrt{3}}\cdot\sqrt{100} = \frac{4}{\sqrt{3}}\cdot 10$ $\quad \frac{4}{\sqrt{3}} \approx 2.4$? So $n > 2.4\cdot 10 = 24$ will work.

(bigger numbers will, too....)