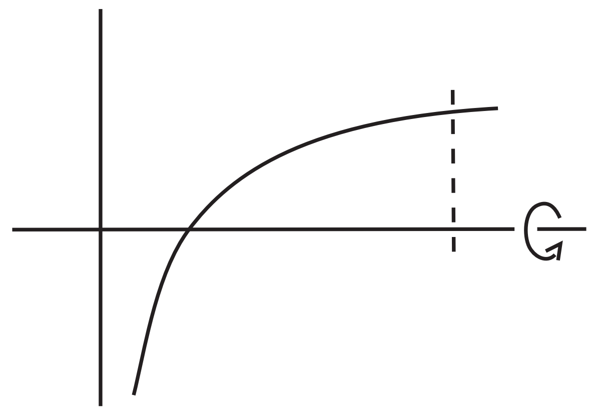
# Math 107H Practice Exam 2 Solutions

**Note:** Most sequences/series can be shown to converge or diverge in more than one way; the solutions below illustrate only one such method. Your approach may differ....

1. Find the volume of the region obtained by revolving the region under the graph of $f(x) = \ln x$ from $x = 1$ to $x = 3$ around the $x$-axis (see figure).



Integrating slices $dx$: Volume$=\pi \int_1^3 (\ln x)^2\ dx = (*)$

By parts: $u = (\ln x)^2$ , $du = \dfrac{2\ln x}{x}\ dx$ , $dv = dx$ , $v = x$

$(*) = \pi x(\ln x)^2\Big|_1^3 - \pi \int_1^3 2\ln x\ dx = (**)$

By parts again: $u = 2\ln x$ , $du = \dfrac{2}{x}\ dx$ , $dv = dx$ , $v = x$

$(**) = \pi x(\ln x)^2\Big|_1^3 - \pi(2x\ln x|Big|_1^3 - \int_1^3 2\ dx) =$

**2.** Find the improper integral $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\ dx$ .

$\displaystyle\int \frac{1}{x(\ln x)^3}\ dx = \int \frac{du}{u^3}\Big|_{u=\ln x}$ via the $u$-substitution $u = \ln x$, so $du = \dfrac{1}{x}\ dx$,

which equals $\displaystyle\int u^{-3}\ du\Big|_{u=\ln x} = -\frac{1}{2}u^{-2} + c\Big|_{u=\ln x} = -\frac{1}{2(\ln x)^2} + c$

So $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\ dx = \lim_{n\to\infty} \int_2^N \frac{1}{x(\ln x)^3}\ dx$

$\displaystyle= \lim_{n\to\infty} -\frac{1}{2(\ln x)^2}\Big|_2^N = \lim_{n\to\infty} \frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln N)^2}$

But since $\ln N \to \infty$ as $N \to \infty$, $\dfrac{1}{2(\ln N)^2} \to 0$ as $N \to \infty$, so

**3.** Determine the convergence or divergence of the following sequences:

**(a)** $a_n = \dfrac{n^3 + 6n^2\ln n - 1}{2 - 3n^3} = \dfrac{1 + 6(\ln n)/n - 1/n^3}{2/n^3 - 3}$.

and since $1/n^3 \to 0$ and $(\ln n)/n \to 0$ as $n \to \infty$,

$$a_n \to \frac{1 + 6 \cdot 0 - 0}{2 \cdot 0 - 3} = \frac{1}{-3} = -\frac{1}{3} \text{ as } n \to \infty.$$

**(b)** $b_n = \dfrac{n^{n+\frac{1}{n}}}{(n+3)^n}$

$$b_n = \frac{n^{n+\frac{1}{n}}}{(n+3)^n} = \frac{n^n n^{\frac{1}{n}}}{(n+3)^n} = \frac{n^{\frac{1}{n}}}{\left(\frac{n+3}{n}\right)^n} = \frac{n^{\frac{1}{n}}}{\left(1+\frac{3}{n}\right)^n}.$$

But $n^{\frac{1}{n}} \to 1$ and $\left(1+\dfrac{3}{n}\right)^n \to e^3$ as $n \to \infty$, so $b_n \to \dfrac{1}{e^3} = e^{-3}$ as $n \to \infty$ .

**4.** Determine the convergence or divergence of the following series:

**(a)** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)(\ln n)^{2/3}}$ [Hint: limit compare, then integral...]

$a_n = \dfrac{1}{(n-1)(\ln n)^{2/3}}$ looks like $b_n = \dfrac{1}{n(\ln n)^{2/3}}$, and $\dfrac{a_n}{b_n} = \dfrac{n}{n-1} \to 1$ as $n \to \infty$, so $\sum a_n$ converges precisely when $\sum b_n$ converges. But:

$b_n = \dfrac{1}{n(\ln n)^{2/3}} = f(n)$ for $f(x) = \dfrac{1}{x(\ln x)^{2/3}}$, which is continuous and decreasing ($x$ and $\ln(x)$ are both increasing, so $(\ln x)^{2/3}$ is increasing, so their reciprocals are decreasing, and so the product is decreasing). So we can apply the integral test:

$\displaystyle\int \frac{1}{x(\ln x)^{2/3}}\,dx = \int \frac{du}{u^{2/3}}\,du|_{u=\ln x} = 3u^{1/3}|_{u=\ln x} = 3(\ln x)^{1/3}$, so

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\,dx = \lim_{N\to\infty}[3(\ln N)^{1/3} - 3(\ln 2)^{1/3}]$, but since $\ln N \to \infty$ as $N \to \infty$,

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\,dx$ diverges, so $\sum b_n$ diverges, so $\sum a_n$ **diverges**.

**(b)** $\displaystyle\sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ $\qquad a_n = \dfrac{6n}{(1-n^2)^2}$ looks like $b_n = \dfrac{6n}{(-n^2)^2} = \dfrac{6}{n^3}$, which converges:

So note that $\dfrac{a_n}{b_n} = \dfrac{(-n^2)^2}{(1-n^2)^2} = \dfrac{1}{(\frac{1}{n^2}-1)^2}$, and since $1/n^2 \to 0$ as $n \to \infty$, $\dfrac{a_n}{b_n} \to \dfrac{1}{(0-1)^2} = 1$ as $n \to \infty$, so by limit comparison, $\sum a_n$ converges precisely when $\sum b_n$ converges.

But: $\sum b_n = \sum \dfrac{6}{n^3} = 6\sum \dfrac{1}{n^3}$, which converges ($p$-series, $p = 3 > 1$), so $\sum b_n$ converges, so $\sum a_n = \displaystyle\sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ **converges**.

**6.** Set up, **but do not evaluate**, the integral which will compute the arclength of the graph of $y = x\sqrt{1+x^2}$ from $x = 0$ to $x = 3$.

$f(x) = x\sqrt{1+x^2} = x(1+x^2)^{\frac{1}{2}}$, so $f'(x) = (1+x^2)^{\frac{1}{2}} + x(\frac{1}{2})(1+x^2)^{-\frac{1}{2}}(2x) = (1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}$.

So Arclength $= \int_0^3 \sqrt{1+[f'(x)]^2}\,dx = \int_0^3 \sqrt{1+[(1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}]^2}\,dx$

6. Find the following limits:

   (a) $\lim_{n\to\infty} \frac{1+\sqrt{2n}}{\sqrt{n}} = (*)$

   $$(*) = \lim_{n\to\infty} \frac{(1/\sqrt{n}) + (\sqrt{2}\sqrt{n}/\sqrt{n})}{(\sqrt{n}/\sqrt{n})} = \lim_{n\to\infty} \frac{(1/\sqrt{n}) + \sqrt{2}}{1}$$
   $$= \lim_{n\to\infty} \sqrt{1/n} + \sqrt{2} = \sqrt{0} + \sqrt{2} = \sqrt{2},$$

   since $a_n = 1/n \to 0$ as $n \to \infty$, so $\sqrt{a_n} \to \sqrt{0}$, since $\sqrt{x} \to 0$ as $x \to 0$ .

   (b) $\lim_{n\to\infty} \frac{4^n + 3^n}{4^n - 3^n} = (**)$

   $$(**) = \lim_{n\to\infty} \frac{4^n/4^n + 3^n/4^n}{4^n/4^n - 3^n/4^n} = \lim_{n\to\infty} \frac{1+(3/4)^n}{1-(3/4)^n} = \frac{1+0}{1-0} = 1,$$

   since $(3/4)^n \to 0$ as $n \to \infty$, since $|3/4| < 1$ .

1. Use a comparision theorem to decide if the following improper integral converges (if yes, you do *not* need to find the value of the integral):

   $$\int_7^\infty \frac{x\ln x}{x^2+1}\,dx$$

   $\frac{x\ln x}{x^2+1}$ 'behaves like' $\frac{x\ln x}{x^2} = \frac{\ln x}{x}$, which is larger than $\frac{1}{x}$, whose integral diverges, so we should try to establish that the integral diverges:
   $\frac{x\ln x}{x^2+1} > \frac{x\ln x}{x^2+x^2} = \frac{x\ln x}{2x^2} = \frac{\ln x}{2x}$.
   Since $\int \frac{\ln x}{2x}\,dx = \int \frac{u}{2}\,du\Big|_{u=\ln x} = \frac{u^2}{4}\Big|_{u=\ln x} = \frac{(\ln x)^2}{4} + C$, and $\frac{(\ln x)^2}{4} \to \infty$ as $x \to \infty$, the integral $\int_7^\infty \frac{\ln x}{2x}\,dx$ diverges. And since $\frac{x\ln x}{x^2} > \frac{\ln x}{2x}$, this means that $\int_7^\infty \frac{x\ln x}{x^2+1}\,dx$ diverges, by comparision.

   Alternatively, we could have continued further, arguing that $\frac{x\ln x}{x^2+1} > \frac{1}{2x}$, and since $\int_7^\infty \frac{1}{2x}\,dx$ (also) diverges, the original integral diveges by comparison.

2. Find the volume of the region obtained by spinning the triangle with sides lying along the lines $y = \frac{1}{2}x$, $x = 4$, and $y = 1$, *around the line* $y = -2$.

Finding the $x$- and $y$-values where the lines cross, the region can be described either as the points lying between $y = 1$ and $y = x/2$ for $2 \leq x \leq 4$, or (writing $y = x/2$ as $x = 2y$) the points lying between $x = 2y$ and $x = 4$ for $1 \leq y \leq 2$.

Treating it the first way, then revolving around $y = -2$ and setting up an integral $dx$ we would use 'slices'/'washers', and integrate the area of a slice, $\pi((x/2)+2)^2 - (1+2)^2)$, from $x = 2$ to $x = 4$. [The '+2' is because the radii (the distance to the axis) is 2 past the $y$-axis.] So we get

Volume $= \int_2^4 \pi(\frac{x}{2}+2)^2 - 3^2)\ dx = \pi(2\frac{1}{3}(\frac{x}{2}+2)^3 - 9x)\Big|_2^4 = \pi[\frac{2}{3}((2+2)^3 - (1+2)^3)) - 9(4-2)] = \pi[\frac{2}{3}(64-27) - 18] = \frac{2\pi}{3}(64-27-27) = \frac{2\pi}{3}(10)$

Alternatively, we could integrate $dy$; this will involve cylcindrical shells. The 'height' is $4 - 2y$, and the radius of $y + 2$ (since we again measure to the line $y = -2$). This gives

Volume $= \int_1^2 2\pi(y+2)(4-2y)\ dy = 2\pi \int_1^2 8 - 2y^2\ dy = 2\pi(8y - \frac{2}{3}y^3)\Big|_1^2 = 2\pi(8(2-1) - \frac{2}{3}(8-1)) = 2\pi(8 - \frac{14}{3}) = 2\pi\frac{10}{3}$

3. Set up, *but do not evaluate*, the integral which evaluates to the length of the spiral, with parametric equation

$$x = t\cos t\ ,\ y = t\sin t\ , \qquad \text{for } 0 \leq t \leq 4\pi\ .$$

Since $x(t) = t\cos t$ and $y(t) = t\sin t$, we have $x'(t) = \cos t - t\sin t$ and $y'(t) = \sin t + t\cos t$. So the speed along the spiral is $\sqrt{(x'(t)^2 + (y'(t))^2} = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$. Then the length of the spiral is

Length $= \displaystyle\int_0^{4\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\ dt.$

Now, as it happens, this is an integral we can do!

$(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2$
$= \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t$
$= (\sin^2 t + \cos^2 t) + t^2(\sin^2 t + \cos^2 t)$
$= 1 + t^2.$

So the integral is really $\int_0^{4\pi} \sqrt{t^2+1}\ dt$, which we could handle by trig substitution.....!

4. Find the limit of each of the following sequences, *if it exists*:

(a) $a_n = \dfrac{2 + \sqrt{n^2 + 5n - 1}}{7n + 12}$

The dominant terms on top and bottom are both $n$ $(= \sqrt{n^2})$, and so dividing we get

$$a_n = \frac{2+\sqrt{n^2+5n-1}}{7n+12} = \frac{\frac{2+\sqrt{n^2+5n-1}}{n}}{\frac{7n+12}{n}} = \frac{\frac{2}{n}+\sqrt{\frac{n^2+5n-1}{n^2}}}{7+\frac{12}{n}} = \frac{\frac{2}{n}+\sqrt{1+\frac{5}{n}-\frac{1}{n^2}}}{7+\frac{12}{n}}$$

Then, since $\frac{1}{n}$, $\frac{2}{n}$, $\frac{5}{n}$, $\frac{12}{n}$ and $\frac{1}{n^2}$ all $\to 0$ as $n \to \infty$, we have
$a_n \to \frac{0+\sqrt{1+0-0}}{7+0} = \frac{1}{7}$ as $n \to \infty$.

(b) $b_n = (n^2+2)^{\frac{1}{n}}$ [Hint: take logs, first!]
[Also, recall *L'Hôpital's Rule*: if $f(x), g(x) \to \infty$ as $x \to \infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \text{ .]}$$

$c_n = \ln(b_n) = \ln((n^2+2)^{\frac{1}{n}}) = \frac{1}{n}\ln(n^2+2) = \frac{\ln(n^2+2)}{n}$.
Then since $f(x) = \ln(x^2+2)$ and $g(x) = x$ both $\to \infty$ as $x \to \infty$, we can use L'Hôpital's Rule: since $f'(x) = \frac{2x}{x^2+2}$ and $g'(x) = 1$, $\frac{f'(x)}{g'(x)} = \frac{2x}{x^2+2} = \frac{2/x}{1+2/x^2} \to \frac{0}{1+0} = 0$ as $x \to \infty$.

So $\frac{\ln(n^2+2)}{n} \to 0$ as $n \to \infty$, and so $(n^2+2)^{\frac{1}{n}} = e^{\ln((n^2+2)^{\frac{1}{n}})} \to e^0 = 1$ as $n \to \infty$.

8. *Use a comparison test* to determine the convergence or divergence of each of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^{\frac{1}{3}}}{\sqrt{n^3+7}}$

Looking at the dominant terms, this series behaves like one with $n$-th term
$\frac{n^{\frac{1}{3}}}{\sqrt{n^3}} = n^{\frac{1}{3}-\frac{3}{2}} = n^{-\frac{7}{6}}$, which converges.

More precisely, $\displaystyle\lim_{n\to\infty} \frac{\frac{n^{\frac{1}{3}}}{\sqrt{n^3+7}}}{\frac{n^{\frac{1}{3}}}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^3+7}{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{7}{n^3}}} = \frac{1}{\sqrt{1+0}} = 1.$

So since $\displaystyle\sum_{n=1}^{\infty} n^{-\frac{7}{6}}$ converges [$p$-series with $p = \frac{7}{6} > 1$],

$\displaystyle\sum_{n=0}^{\infty} \frac{n^{\frac{1}{3}}}{\sqrt{n^3+7}}$ converges, by the limit comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$

Looking at the dominant terms, this series behaves like one with $n$-th term $\frac{2^n}{n^2 2^n} = \frac{1}{n^2}$, which converges.

More precisely, $\displaystyle\lim_{n\to\infty} \frac{\left(\frac{n+2^n}{n^2 2^n}\right)}{\left(\frac{2^n}{n^2 2^n}\right)} = \lim_{n\to\infty} \frac{\frac{n}{2^n}+1}{1} = \lim_{n\to\infty} \frac{n}{2^n} + 1 = 0 + 1 = 1$, since

$\frac{n}{2^n} \to 0$ as $n \to \infty$, by L'Hôpital:

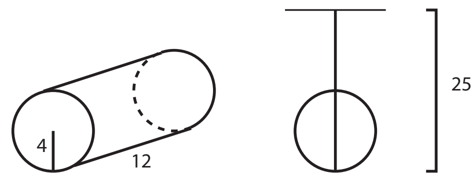
$$\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{(x)'}{(2^x)'} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0,$$ since $2^x$ gets (really) large as $x$ gets large.

So since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges [$p$-series with $p = 2 > 1$],

$\sum_{n=0}^{\infty} \frac{n+2^n}{n^2 2^n}$ converges, by the limit comparison test.

10. A cylindrical tank is buried on its side so that its bottom is 25 ft deep (see figure). The 'height' of the cyclinder is 12 ft and the 'radius' of the cylinder is 4 ft. If the cyclinder is full of oil, having a weight-density of 50 lb/ft$^3$, set up, but do not evaluate, the integral which will compute how much work is needed to lift the oil to the surface.

(Note: weight-density is force per cubic foot!)




To set up the integral, we need coordinates, and there are lots of choices. If we set $x = 0$ to be the center of the circle, then the surface is at $x = 21$. Cross sections of the cylinder are rectangles, with length 12 and width the width of the circle at 'height' $x$, which since the circle is given by the equation (note that '$x$' is the 'second' coordinate!) $a^2 + x^2 = 4^2 = 16$, has width $= 2a = 2\sqrt{4^2 - x^2}$. So we need to lift the volume $(12)(2\sqrt{4^2 - x^2})\ dx$ a distance $21 - x$, where $x$ runs from $-4$ to 4. With a weight-density of 50, the work done is then

$$\int_{-4}^{4} [(50)(12)(2\sqrt{4^2 - x^2})][21 - x]\ dx = \int_{-4}^{4} 1200(21 - x)\sqrt{16 - x^2}\ dx$$

If we instead put $x = 0$ at ground level, we will lift slices a distance $-x$, where $x$ runs from $-25$ to $-17$. The circle is then the circle of radius 4 with center $(0, -21)$ (note that the second coordinate is the $x$-coord!), so has equation $a^2 + (x + 21)^2 = 16$, so $a = \pm\sqrt{16 - (x + 21)^2}$. So the cross-section of the cylinder has area $(12)(2\sqrt{16 - (x + 21)^2})$, giving us the integral

$$\int_{-25}^{-17} [(50)(12)(2\sqrt{16 - (x + 21)^2})][-x]\ dx = \int_{-25}^{-17} -1200x\sqrt{16 - (x + 21)^2)}\ dx$$

which is the same integral as the first, after the $u$-substitution $u = x - 21$ ...!

Or! You might set $x = 0$ at the bottom of the tank, in which case you integrate from $x = 0$ to $x = 8$, your cross-sectional area is $(12)(2\sqrt{16 - (x - 4)^2})$, and you lift that slice a distance $25 - x$. This gives the integral

$$\int_{0}^{8} [(50)(12)(2\sqrt{16 - (x - 4)^2})][25 - x]\ dx = \int_{0}^{8} 1200(25 - x)\sqrt{16 - (x - 4)^2)}\ dx$$

which, again, is the same integral, after the $u$-substitution $u = x + 4$ .

Of all of these, the first is probably the least painful one to actually integrate. Splitting it up,

$$\int_{-4}^{4} [1200(21-x)\sqrt{16-x^2}\ dx = \int_{-4}^{4} [(1200)(21)\sqrt{16-x^2}\ dx - \int_{-4}^{4} 1200x\sqrt{16-x^2}\ dx$$

But!

$$\int_{-4}^{4} \sqrt{16-x^2}\ dx$$

is the area of a semicircle of radius 4 (i.e, $(1/2)\pi(4)^2 = 8\pi$), and

$$\int_{-4}^{4} x\sqrt{16-x^2}\ dx$$

is the integral of an odd function over the interval $[-4, 4]$, and so is 0 (!). So the total work done is $(1200)(21)(8\pi)$ foot-pounds.