Old Final solutions

1-1: $\int \sec^3 x \tan^3 x\,dx = \int \sec^2 x \tan^2 x\,(\sec x \tan x\,dx)$

$= \int \sec^2 x(\sec^2 x - 1)(\sec x \tan x\,dx)$ $\quad [u = \sec x \quad du = \sec x \tan x\,dx]$

$= \int u^2(u^2-1)\,du \Big|_{u=\sec x} = \int u^4 - u^2\,du \Big|_{u=\sec x} = \frac{u^5}{5} - \frac{u^3}{3} + C \Big|_{u=\sec x}$

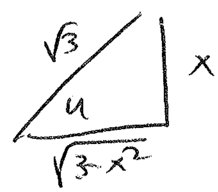
$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$.

1-2: $\int \frac{x^2\,dx}{\sqrt{3-x^2}}$ $\quad x = \sqrt{3}\sin u,\ dx = \sqrt{3}\cos u\,du,\ 3 - x^2 = 3\cos^2 u$

$= \int \frac{(\sqrt{3}\sin u)^2(\sqrt{3}\cos u\,du)}{\sqrt{3\cos^2 u}} \Big|_{x=\sqrt{3}\sin u} = \int 3\sin^2 u\,du \Big|_{x=\sqrt{3}\sin u}$

$= 3\int \frac{1}{2}(1-\cos 2u)\,du \Big|_{x=\sqrt{3}\sin u} = \frac{3}{2}\left(u - \frac{1}{2}\sin 2u + C\right)\Big|_{x=\sqrt{3}\sin u}$

$= \frac{3}{2}(u - \sin u \cos u) + C \Big|_{x=\sqrt{3}\sin u}$



$\boxed{= \frac{3}{2}\left(\mathrm{Arcsin}\left(\frac{x}{\sqrt{3}}\right) - \frac{x}{\sqrt{3}}\frac{\sqrt{3-x^2}}{\sqrt{3}}\right) + C}$.

1-3: $\int x^2 e^{3x}\,dx$ $\quad u = x^2,\ dv = e^{3x}dx \quad du = 2x\,dx,\ v = \frac{1}{3}e^{3x}$

$\quad u = x \quad dv = e^{3x} \quad du = dx \quad v = \frac{1}{3}e^{3x}$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x}\,dx$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x}\,dx\right) = \boxed{\frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C}$

1-4: $\int \frac{2x+3}{x^3+x^2-2}\,dx = \int \frac{2x+3}{(x-1)(x^2+2x+2)}\,dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}\,dx$

$2x+3 = A(x^2+2x+2) + (x-1)(Bx+C)$: $\quad x=1,\ 5 = A(5) \to A = 1$

$x=0,\quad 3 = (1)(2) + (-1)(C),\quad C = 2-3 = -1,\quad x=-1,\quad 1 = (1)(1) + (-2)(-B-1)$

$-B-1 = 0,\quad B = -1$

$$\int \frac{2x+3}{x^3+x^2-2}\,dx = \int \frac{1}{x-1} - \frac{x+1}{x^2+2x+2}\,dx = \ln|x-1| - \int \frac{x+1}{(x+1)^2+1}\,dx$$

$$= \ln|x-1| - \int \frac{u}{u^2+1}\,du \Big|_{u=x+1} = \ln|x-1| - \int \frac{\frac{1}{2}dv}{v}\Big|_{v=u^2+1}\Big|_{u=x+1}$$

$$= \ln|x-1| - \frac{1}{2}\ln|v| + C\Big|_{v=u^2+1}\Big|_{u=x+1} = \ln|x-1| - \frac{1}{2}\ln|u^2+1| + C\Big|_{u=x+1}$$

$$\boxed{= \ln|x-1| - \ln\left|(x+1)^2+1\right| + C}$$

$x^4 - x = 0 = (x^3-1)x$

2. $f(x)=g(x)$: $2x-1 = x^4+x-1$,

$x^3=1 \to x=1$

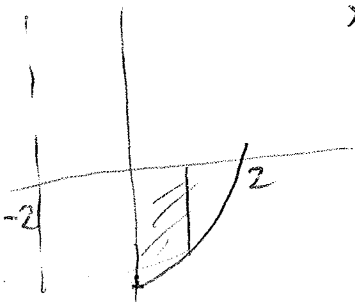
$\Rightarrow x=0$ or $x^3-1=0 \to$

on $[0,1]$, $2x-1 \geq x^4+x-1$ (check $x=\frac{1}{2}$: $0=1-1 > \frac{1}{16}+\frac{1}{2}-1$)

$$\text{Area} = \int_0^1 (2x-1)-(x^4+x-1)\,dx = \int_0^1 -x^4+x\,dx$$

$$= \frac{x^2}{2}-\frac{x^5}{5}\Big|_0^1 = \left(\frac{1}{2}-\frac{1}{5}\right)-(0-0) = \frac{1}{2}-\frac{1}{5} = \frac{5}{10}-\frac{2}{10} = \boxed{\frac{3}{10}}$$

$x^3+7x-22=0$: $(x-2)(x^2+2x+11)=0$ $\underline{x=2}$.

3. 

By shells: (width) (height)

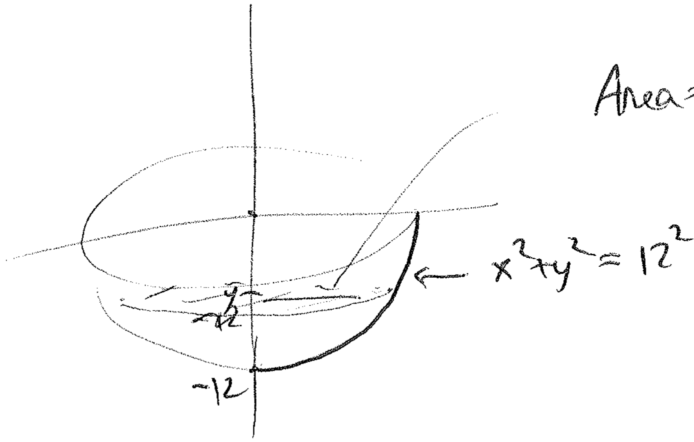
$$\text{Volume} = \int_0^2 2\pi(x-(-2))(x^3+7x-22)\,dx$$

$$= \int_0^2 2\pi(x+2)(x^3+7x-22)\,dx = 2\pi\int_0^2 x^4+2x^3+7x^2+14x-22x-44\,dx$$

$$= 2\pi\int_0^2 x^4+2x^3+7x^2-8x-44\,dx = 2\pi\left(\frac{x^5}{5}+\frac{x^4}{2}+\frac{7x^3}{3}-4x^2-44x\right)\Big|_0^2$$

$$= \boxed{2\pi\left(\left(\frac{2^5}{5}+\frac{2^4}{2}+\frac{7\cdot 8}{3}-4\cdot 4-44\cdot 2\right)-(0)\right)}$$

4.



Area $= A(x) = \pi x^2 = \pi(144-y^2)$

$$\text{work} = \int_{-12}^{0} \cancel{\pi k}\underbrace{(-y)}_{\text{distance}}\underbrace{\underset{300}{\rho}\,\pi(144-y^2)\,dy}_{\text{mass}}$$

$$= 300\pi\int_{-12}^{0} -144y+y^3\,dy = 300\pi\left(-72y^2+\frac{y^4}{4}\right)\Big|_{-12}^{0}$$

$$= 300\pi\left((0+0)-\left(-72(-12)^2+\frac{(-12)^4}{4}\right)\right)$$

$$= 300\pi\left(72\cdot 12^2 - \frac{12^2}{4}\cdot 12^2\right) = 300\pi\cdot 12^2(72-36)$$

$$= \boxed{300\pi\cdot 12^2\cdot 36.}$$

5 (a) $$\lim_{x\to\infty}\frac{x^2-3x^3+9}{4x^2-6x+1} = \lim_{x\to\infty}\frac{1-\overset{\to -\infty}{3x}+\overset{\to 0}{9/x^2}}{4-\underset{\to 0}{6\,1/x}+\underset{\to 0}{1/x^2}} \approx \frac{\text{large neg}}{4} = \boxed{-\infty}$$

(b) $$\lim_{x\to\infty}\frac{(x^2+1)^x}{(x+1)^{2x}} = L \qquad \ln L = \lim_{x\to\infty} x\ln(x^2+1)-2x\ln(x+1)$$

$$= \lim x(\ln(x^2+1)-2\ln(x+1)) = \lim x\ln\left(\frac{x^2+1}{(x+1)^2}\right) = \lim x\ln\left(\frac{1+(\frac{1}{x})^2}{(1+(\frac{1}{x}))^2}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{1+(\frac{1}{x})^2}{(1+(\frac{1}{x}))^2}\right)}{1/x}$$

$$= \lim_{h\to 0^+}\frac{1}{h}\ln\left(\frac{1+h^2}{(1+h)^2}\right) = f'(0), \quad f(x) = \ln\left(\frac{1+x^2}{(1+x)^2}\right)$$

But: $$f'(x) = \left(\frac{1}{\left(\frac{1+x^2}{(1+x)^2}\right)}\right)\left(\frac{(1+x)^2(2x)-(1+x^2)(2(1+x))}{(1+x)^2}\right);\ \text{at } x=0,$$

$$f'(0) = \frac{1}{\left(\frac{1}{1^2}\right)}\left(\frac{(1)(0)-(1)(2)}{1^2}\right) = -2. \quad \text{So } \ln L = -2, \text{ so } \boxed{L = e^{-2}}.$$

6-1: $\sum_{n=1}^{\infty} \frac{(n+1)^{1/2}}{n^2} = \sum a_n$ $\quad b_n = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$ then

$\frac{a_n}{b_n} = \frac{(n+1)^{1/2}}{n^{1/2}} = \left(1+\frac{1}{n}\right)^{1/2} \to 1 \neq 0$ as $n\to\infty$, so since

$\sum b_n$ converges (p-series, $p = 3/2 > 1$), $\boxed{\sum a_n \text{ conv}}$ by lim. compar.

6-2: $\sum \frac{n!}{(n^2+n-3)^{3/2}} = \sum a_n$ $\quad b_n = \frac{n!}{(n^2)^{3/2}} = \frac{n!}{n^3}$ then

$\frac{a_n}{b_n} = \left(\frac{n^2}{n^2+n-3}\right)^{3/2} \to 1 \neq 0$ as $n\to\infty$. But $\sum \frac{n!}{n^3} \geq \sum \frac{n(n-1)(n-2)(n-3)}{n^3}$

and $\frac{n(n-1)(n-2)}{n^3} = (1-\frac{1}{n})(1-2\frac{1}{n}) \to 1 \neq 0$ as $n\to\infty$ so $\sum b_n$

diverges by n^{th} term test, so $\boxed{\sum a_n \text{ diverges}}$ by lim compar.

6-3: $\sum \left(\frac{n+3}{3n-5}\right)^n = \sum a_n$ $\quad a_n^{1/n} = \frac{n+3}{3n-5} = \frac{1+3\frac{1}{n}}{3-5\frac{1}{n}} \to \frac{1}{3} < 1$

as $n\to\infty$ so $\boxed{\sum a_n \text{ conv}}$ by the root test.

6-4: $\sum_{n=1}^{\infty} \frac{\ln n}{n^{5/3}} = \sum a_n$. But $\frac{\ln n}{n^\alpha} \to 0$ as $n\to\infty$ for any

$\alpha > 0$, so compare to $\quad b_n = \frac{n^{1/3}}{n^{5/3}} = \frac{1}{n^{4/3}}$.

$\frac{a_n}{b_n} = \frac{\ln n}{n^{1/3}} \to 0$ as $n\to\infty$ and since $\sum b_n$ conv (p-series, $p = 4/3 > 1$) $\boxed{\sum a_n \text{ conv}}$ by limit comparison.

7: $f(x) = (x^2-5)^{5/2}$ centered at $c=3$. $\quad f(3) = (9-5)^{5/2} = 4^{5/2} = 2^5 = 32$.

$f'(x) = \frac{5}{2}(x^2-5)^{3/2}(2x)$ $\quad f'(3) = \frac{5}{2}(4)^{3/2}(6) = 5\cdot 2^3 \cdot 3 = 120$

$f''(x) = \frac{5}{2}\left(\frac{3}{2}(x^2-5)^{1/2}(2x)(2x) + 2(x^2-5)^{3/2}\right)$

$f''(3) = \frac{5}{2}\left(\frac{3}{2}4^{1/2}(6)(6) + 2(4)^{3/2}\right) = \frac{5}{2}\left(108 + 16\right) = 310$ $\quad \begin{matrix}62\\124\end{matrix}$

$$f'''(x) = \frac{5}{2}\left(\frac{3}{2}\left(\frac{1}{2}(x^2-5)^{-1/2}(2x)(4x^2) + (8x)(x^2-5)^{1/2}\right) + 2\left(\frac{3}{2}\right)(x^2-5)^{1/2}(2x)\right)$$

$$f'''(3) = \frac{5}{2}\left(\frac{3}{2}\left(\frac{1}{2}(4)^{-1/2}(6)(4\cdot 6^2) + (24)(4)^{1/2}\right) + 3(4)^{1/2}(6)\right)$$

$$= \frac{5}{2}\left(\frac{3}{2}(\overset{264}{216+48}) + 36\right) = \frac{5}{2}\left(\underset{396}{3\cdot 132} + 36\right) = \frac{5}{2}(\underset{216}{\cancel{432}}) = 1080.$$

So $$P_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2 + \frac{f'''(3)}{6}(x-3)^3$$

$$\boxed{= 32 + 120(x-3) + 155(x-3)^2 + 180(x-3)^3}$$

8: $x(t) = t^4$, $y(t) = t^6$ $\quad x'(t) = 4t^3 \quad y'(t) = 6t^5$

$$\text{Length} = \int_0^2 \sqrt{(4t^3)^2 + (6t^5)^2}\, dt = \int_0^2 \sqrt{16t^6 + 32t^{10}}\, dt$$

$$= \int_0^2 t^3 (16 + 32t^4)^{1/2}\, dt \qquad u = 16 + 32t^4 \quad du = 128t^3\,dt$$

$$t^3\,dt = \frac{1}{128}\,du$$

$$t=0 \to u = 16$$

$$t=2 \to u = 16 + 32\cdot 16 = 33\cdot 16 = 480+48 = 528$$

$$= \int_{16}^{528} u^{1/2} \frac{1}{128}\, du$$

$$= \frac{1}{128}\frac{2}{3}u^{3/2}\Big|_{16}^{528} = \boxed{\frac{1}{3\cdot 64}\left((528)^{3/2} - 16^{3/2}\right)}$$

5(b), REDUX: $$\frac{(x^2+1)^x}{(x+1)^{2x}} = \left(\frac{x^2+1}{(x+1)^2}\right)^x = \left(\frac{x^2+1}{x^2+2x+1}\right)^x = \left(1 - \frac{2x}{(x+1)^2}\right)^x$$

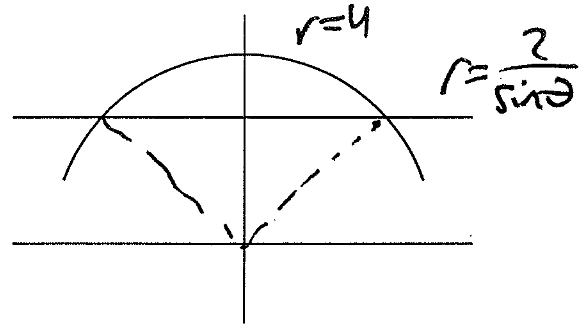
$$= \left(1 - \frac{2}{\left(\frac{(x+1)^2}{x}\right)}\right)^x = \left(\left(1 - \frac{2}{\left(\frac{(x+1)^2}{x}\right)}\right)^{\left(\frac{(x+1)^2}{x}\right)}\right)^{\frac{x^2}{(x+1)^2}} = \left(\left(1 - \frac{2}{\text{blah}}\right)^{\text{blah}}\right)^{\left(\frac{x}{x+1}\right)^2}$$

But! blah $\to \infty$ as $x \to \infty$, $\left(1 - \frac{2}{\text{blah}}\right)^{\text{blah}} \to e^{-2}$ as blah $\to \infty$, and $\left(\frac{x}{(x+1)}\right)^2 \to 1$ as $x \to \infty$, so as $x \to \infty$

$$\frac{(x^2+1)^x}{(x+1)^{2x}} = \left(\left(1 + \frac{-2}{\text{blah}}\right)^{\text{blah}}\right)^{\left(\frac{x}{x+1}\right)^2} \to (e^{-2})^1 = e^{-2}$$

6. (15 pts.) Find the area lying between the polar curves $r = 4$ and $r = \dfrac{2}{\sin(\theta)}$ (see figure).

pts of intersection:



$$4 = \frac{2}{\sin\theta} \qquad \sin\theta = \frac{2}{4} = \frac{1}{2}$$

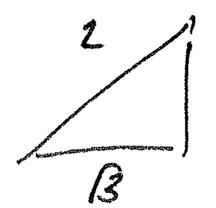
$$\theta = \frac{\pi}{6},\ \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$\frac{2}{\sin\theta}$ is inside of 4 in this range, so:

$$\text{Area} = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4)^2 - \frac{1}{2}\left(\frac{2}{\sin\theta}\right)^2 d\theta = \int_{\pi/6}^{5\pi/6} 8 - \frac{2}{\sin^2\theta}\, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 8 - 2\csc^2\theta\, d\theta = 8\theta - 2(-\cot\theta)\Big|_{\pi/6}^{5\pi/6}$$

$$= 8\theta + 2\cot\theta\Big|_{\pi/6}^{5\pi/6} = \left(8\frac{5\pi}{6} + 2\cot\left(\frac{5\pi}{6}\right)\right) - \left(8\left(\frac{\pi}{6}\right) + 2\cot\left(\frac{\pi}{6}\right)\right)$$



$$= \left(\frac{20\pi}{3} + 2(-\sqrt{3})\right) - \left(\frac{4\pi}{3} + 2(\sqrt{3})\right)$$

$$= \frac{16\pi}{3} - 4\sqrt{3}$$

4. For the integrals below, when the appropriate substitution is made, what (trigonometric) integral results? **Express your integrand in terms of** $\sin x$ **and** $\cos x$.

(a) (10 pts.) $\int \frac{\sqrt{x^2-2}}{x^2}\,dx$

$x = \sqrt{2}\sec u \quad dx = \sqrt{2}\sec u \tan u\,du$

$x^2 - 2 = 2\sec^2 u - 2 = 2\tan^2 u$

$$= \int \frac{\sqrt{2}\tan u}{(\sqrt{2}\sec u)^2}\sqrt{2}\sec u \tan u\,du\Big|_{x=\sqrt{2}\sec u} = \int \frac{\tan^2 u}{\sec u}\,du\Big|_{x=\sqrt{2}\sec u}$$

$$= \int \frac{\frac{\sin^2 u}{\cos^2 u}}{\frac{1}{\cos u}}\,du\Big|_{x=\sqrt{2}\sec u} = \int \frac{\sin^2 u}{\cos u}\Big|_{x=\sqrt{2}\sec u}.$$

(b) (10 pts.) $\int \frac{x^2}{\sqrt{3-x^2}}\,dx$

$x = \sqrt{3}\sin u \quad dx = \sqrt{3}\cos u\,du$

$3 - x^2 = 3 - 3\sin^2 u = 3\cos^2 u$

$$= \int \frac{(\sqrt{3}\sin u)^2}{\cancel{\sqrt{3}}\cancel{\cos u}}\cancel{\sqrt{3}}\cancel{\cos u}\,du\Big|_{x=\sqrt{3}\sin u} = \int 3\sin^2 u\,du\Big|_{x=\sqrt{3}\sin u}$$

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6. (15 pts.) Set up, but do not solve (because you can't!), an integral which will compute the length of the ellipse given by the equation

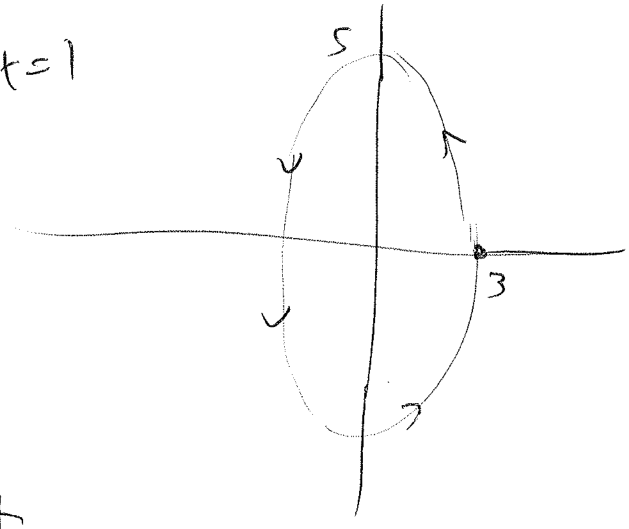
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

[Hint: Finding a parametrization "close to" $x = \cos t$, $y = \sin t$ will help...]

$\sin^2 t + \cos^2 t = 1$

$\frac{x}{3} = \sin t$

$\frac{y}{5} = \cos t$



$x = 3 \sin t$

$y = 5 \cos t$

$0 \le t \le 2\pi$

$x' = 3 \cos t \quad y' = -5 \sin t$

$$\text{length} = \int_0^{2\pi} \left((3 \cos t)^2 + (-5 \sin t)^2 \right)^{1/2} dt$$

$$= \int_0^{2\pi} \left(9 \cos^2 t + 25 \sin^2 t \right)^{1/2} dt$$

$$= \int_0^{2\pi} \left(9 + 16 \sin^2 t \right)^{1/2} dt$$

02

9. (15 pts.) Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n}{n2^n + 3^n}(x-1)^n$$

$a_n = \frac{n}{n2^n+3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{(n+1)2^{n+1}+3^{n+1}} \cdot \frac{n2^n+3^n}{n}$$

$$= \frac{n+1}{n} \cdot \frac{n\left(\frac{2}{3}\right)^n + 1}{2(n+1)\left(\frac{2}{3}\right)^n + 3} \to 1 \cdot \frac{0+1}{0+3} = \frac{1}{3}$$

So $L = \frac{1}{3}$, so $R = \frac{1}{L} = 3$.

$|x-1| < 3$ converges, $|x-1| > 3$ diverges

$x-1 = 3$ $\quad x = -2 \quad x = 4$

$$\sum \frac{n}{n2^n+3^n}(-3)^n = \sum (-1)^n \frac{n3^n}{n2^n+3^n}$$

$$\frac{n3^n}{n2^n+3^n} = \frac{n}{n\left(\frac{2}{3}\right)^n+1} \to \infty$$ so nth term test $\Rightarrow$ diverges

$x = 4$ $\quad \sum \frac{n}{n2^n+3^n}(3)^n = \sum \frac{n3^n}{n2^n+3^n}$ nth term test $\Rightarrow$ diverges

So interval of convergence is $(-2, 4)$.

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10. (15 pts.) Starting from the Taylor series for $f(x) = \dfrac{1}{1-x}$ centered at $a = 0$, show how to build (by multiplication, substitution, differentiation, and/or integration) the Taylor series for the function

$$g(x) = \frac{\ln(1+x^3)}{x}$$

(also) centered at $a = 0$.

[Hint: start by building $h(x) = \dfrac{1}{1+x}$ (!).]

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

So:

$$\ln(1+x) = \int \frac{dx}{1+x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{(integration term-by-term)}$$

So:

$$\ln(1+x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{n+1}$$

So!!

$$\frac{\ln(1+x^3)}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{3n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{3n+2} .$$

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or: $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$; $\dfrac{1}{1+x^3} = \displaystyle\sum_{n=0}^{\infty} (-1)^n (x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$

$$\frac{3x^2}{1+x^3} = (3x^2) \sum (-1)^n x^{3n} = 3 \sum_{n=0}^{\infty} (-1)^n x^{3n+2} \text{; integrate!}$$

$$\ln(1+x^3) = 3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{3n+3} \text{ ; } \frac{\ln(1+x^3)}{x} = \frac{3}{x} \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{3n+3}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{n+1}$$

2-1: $\displaystyle\int (2x-3)^{5/2}\,dx \;\Bigg|\; \begin{array}{l} \text{Set } u=2x-3 \\ du=2dx \\ dx=\tfrac{1}{2}du \end{array} \;\Bigg|\; = \int u^{5/2}\,\tfrac{1}{2}\,du \Big|_{u=2x-3}$

$$= \frac{1}{2}\cdot\frac{2}{7}u^{7/2}+C\Big|_{u=2x-3} = \frac{1}{7}(2x-3)^{7/2}+C$$

2-2: $\displaystyle\int \frac{x\,dx}{(x+1)(x+3)} \overset{(*)}{=} \quad \frac{x}{(x+1)(x+3)} = \frac{A}{x+1}+\frac{B}{x+3} = \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$

$x = A(x+3)+B(x+1)$ $\quad x=-1:\ -1=A(2)\quad A=-\tfrac{1}{2}$

$\quad x=-3:\ -3=B(-2)\quad B=\tfrac{3}{2}$

$$(*) = \int -\frac{1}{2}\frac{dx}{x+1} + \int \frac{3}{2}\frac{dx}{x+3} = -\frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x+3| + C$$

2-3: $\displaystyle\int_0^{\pi} x\sin(2x)\,dx \overset{(*)}{=}$ $\quad u=x \quad du=dx$

$dv=\sin(2x)dx$

$v=-\tfrac{1}{2}\cos(2x)$

$$(*) = x\left(-\tfrac{1}{2}\cos(2x)\right)\Big|_0^{\pi} - \int_0^{\pi} -\tfrac{1}{2}\cos(2x)\,dx$$

$$= -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{2}\left(\tfrac{1}{2}\sin(2x)\right)\Big|_0^{\pi} = \left(\left(-\tfrac{\pi}{2}\right)(1)+\tfrac{1}{4}(0)\right) - \left((0)(1)+\tfrac{1}{4}(0)\right)$$

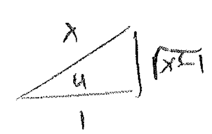
$$= -\pi/2$$

2-4: $\displaystyle\int \sqrt{x^2-1}\,dx \overset{(*)}{=}$ $\quad x=\sec u \qquad \sqrt{x^2-1} = \sqrt{\sec^2 u-1} = \sqrt{\tan^2 u} = \tan u$

$dx = \sec u\tan u\,du$

$$(*) = \int \sec u\tan^2 u\,du\Big|_{x=\sec u} = \int \sec^3 u - \sec u\,du\Big|_{x=\sec u}$$

$$= \tfrac{1}{2}\sec u\tan u\Big|_{x=\sec u} + \tfrac{1}{2}\int \sec u\,du\Big|_{x=\sec u} - \int \sec u\,du\Big|_{x=\sec u} = \left(\tfrac{1}{2}\sec u\tan u - \tfrac{1}{2}\int \sec u\,du\right)\Big|_{x=\sec u}$$



$\uparrow$

reduction formula!

$$= \tfrac{1}{2}\sec u\tan u - \tfrac{1}{2}\ln|\sec u+\tan u| + C\Big|_{x=\sec u}$$

$$= \tfrac{1}{2}x\sqrt{x^2-1} - \tfrac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C$$

2-5. $x(t)=t^2$ $y(t)=t^3$ $0\le t\le 2$ $x'(t)=2t$ $y'(t)=3t^2$

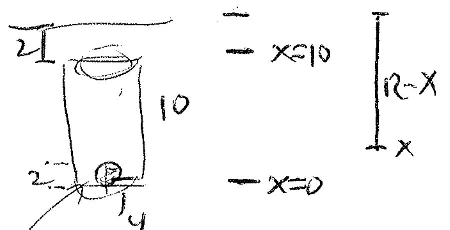
Length $=\int_0^2 \left((2t)^2+(3t^2)^2\right)^{1/2}dt = \int_0^2 (4t^2+9t^4)^{1/2}dt$

$=\int_0^2 (t^2)^{1/2}(4+9t^2)^{1/2}dt = \int_0^2 t(4+9t^2)^{1/2}dt = (*)$

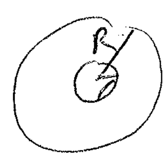
$u=4+9t^2$, $du=18t\,dt$, $t\,dt=\frac{1}{18}du$ $t=0\to u=4$, $t=2\to u=4+9\cdot 4=40$

$(*)=\int_4^{40}\frac{1}{18}u^{1/2}du = \frac{1}{18}\frac{2}{3}u^{3/2}\Big|_4^{40} = \frac{1}{27}\left(40^{3/2}-4^{3/2}\right)$

2-6.



cross sections: $0\le x\le 2$



$R=4$ $r=\sqrt{1-(x-1)^2}$

$2\le x\le 10$

$R=4$

$r^2+(x-1)^2=1$

Work $=\int_0^2 9.8W A(x)(12-x)dx + \int_2^{10} 9.8W A(x)(12-x)dx$

$=9.8W\int_0^2 \pi\left(4^2-(1-(x-1)^2)\right)(12-x)\,dx$

$+9.8W\int_2^{10}\pi(4)^2(12-x)\,dx$.

2-7. $\int_2^\infty \frac{\ln x}{x^2}dx$: $(*)=\int\frac{\ln x}{x^2}dx$ $u=\ln x$ $dv=\frac{1}{x^2}dx$ $du=\frac{1}{x}dx$ $v=-\frac{1}{x}dx$

$(*)=-\frac{\ln x}{x}-\int -\frac{1}{x^2}dx = -\frac{\ln x}{x}+\int\frac{dx}{x^2} = -\frac{\ln x}{x}-\frac{1}{x}+C$

So $\int_2^\infty \frac{\ln x}{x^2}dx = \lim_{N\to\infty} -\frac{\ln x}{x}-\frac{1}{x}\Big|_2^N = \lim_{N\to\infty}\left(\frac{\ln 2}{2}+\frac{1}{2}\right)-\left(\underbrace{\frac{\ln N}{N}}_{\to 0}+\underbrace{\frac{1}{N}}_{\to 0}\right)$

$=\frac{\ln 2}{2}+\frac{1}{2}<\infty$, so integral converges. Since $\left(\frac{\ln x}{x^2}\right)'=\frac{\frac{1}{x}\cdot x^2-\ln x(2x)}{x^4}$

$=\frac{1-2\ln x}{x^3}<0$ (eventually), by the Integral Test, $\sum\frac{\ln n}{n^2}=\sum f(n)$ converges.

2-8 (a) $\sum \frac{n!}{4^n(n+2)^2} = \sum a_n$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{4^{n+1}(n+3)^2} \cdot \frac{4^n(n+2)^2}{n!} = (n+1)\frac{1}{4}\left(\frac{n+2}{n+3}\right)^2 \longrightarrow \underline{\underline{\infty}}$. So

by the Ratio Test, $\sum a_n$ diverges.

2-8(b) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n^2+1} = \sum_{n=0}^{\infty} (-1)^n b_n$ $\quad b_n \geq 0 \quad b_n = f(n)$ for

$f(x) = \frac{x}{x^2+1}$, and $f'(x) = \frac{\cancel{x}(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$ (for $x>1$)

so f is $\searrow$, so ~~an~~ b_n is decreasing, and

$\frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}} \longrightarrow 0$ as $n \to \infty$, so by the Alternating

series test, $\sum (-1)^n b_n$ converges.

2-9 $(*) = \sum \frac{(x+1)^n}{3^n n^2} = \sum a_n (x+1)^n$ for $a_n = \frac{1}{3^n n^2}$

$\frac{a_{n+1}}{a_n} = \frac{3^n n^2}{3^{n+1}(n+1)^2} = \frac{1}{3}\left(\frac{n}{n+1}\right)^2 \to \frac{1}{3} = L$ so $R = \frac{1}{L} = 3 =$ radius of conv.

when $x+1=3$ $(x=2)$, $(*) = \sum \frac{3^n}{3^n n^2} = \sum \frac{1}{n^2}$ which converges (p-series $p=2>1$)

when $x+1=-3$ $(x=-4)$, $(*) = \sum \frac{(-3)^n}{3^n n^2} = \sum \frac{(-1)^n}{n^2}$ which converges (absolutely!)

So $(*)$ converges for $-4 \leq x \leq 2$.

2-10: $f(x) = xe^{-x}$ $\quad P_3(x)$ centered at $a=1$:

$f'(x) = e^{-x} - xe^{-x}$, $f''(x) = -e^{-x} - (e^{-x} - xe^{-x}) = xe^{-x} - 2e^{-x}$

$f'''(x) = e^{-x} - xe^{-x} + 2e^{-x} = 3e^{-x} - xe^{-x}$

So $f(1) = 1 \cdot e^{-1} = e^{-1}$, $f'(1) = e^{-1} - 1 \cdot e^{-1} = 0$ $\quad f''(1) = (1)e^{-1} - 2e^{-1} = -e^{-1}$

$f'''(1) = 3e^{-1} - (1)e^{-1} = 2e^{-1}$, so

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= e^{-1} + 0(x-1) + \frac{-e^{-1}}{2}(x-1)^2 + \frac{2e^{-1}}{6}(x-1)^3$$

$$= e^{-1} + \frac{1}{2}e^{-1}(x-1)^2 + \frac{1}{3}e^{-1}(x-1)^3.$$