The First Half of Calculus
in 10 (or 13) pages

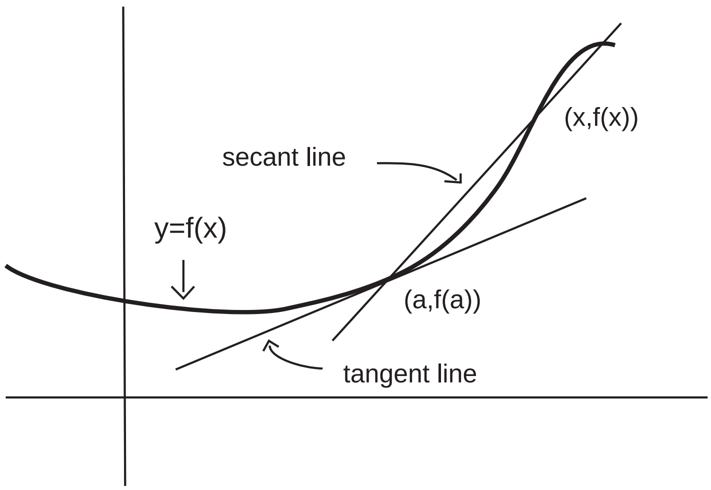
Limits and Continuity

Rates of change and limits:

Limit of a function f at a point a = the value the function 'should' take at the point
= the value that the points 'near' a tell you f should have at a

$\lim\limits_{x\to a} f(x) = L$ means $f(x)$ is close to L when x is close to (but not equal to) a

Idea: slopes of tangent lines



$\lim\limits_{x\to a} f(x) = L$ does not care what $f(a)$ is; it ignores it

$\lim\limits_{x\to a} f(x)$ need not exist! (function can't make up it's mind?)

Rules for finding limits:

If two functions $f(x)$ and $g(x)$ agree (are equal) for every x near a
(but maybe not at a), then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$

Ex.: $\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{x^2-4} = \lim\limits_{x\to 2} \dfrac{(x-1)(x-2)}{(x+2)(x-2)} = \lim\limits_{x\to 2} \dfrac{x-1}{x+2} = \dfrac{1}{4}$

If $f(x) \to L$ and $g(x) \to M$ as $x \to a$ (and c is a constant), then
$f(x)+g(x) \to L+M$; $f(x)-g(x) \to L-M$; $cf(x) \to cL$;
$f(x)g(x) \to LM$; and $f(x)/g(x) \to L/M$ provided $M \neq 0$

If $f(x)$ is a polynomial, then $\lim\limits_{x\to x_0} f(x) = f(x_0)$

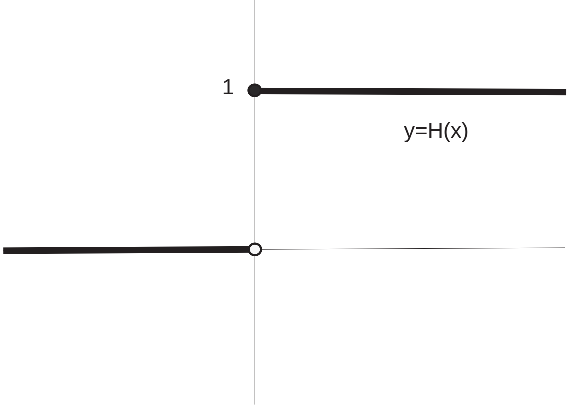
Basic principle: to solve $\lim\limits_{x\to x_0}$, plug in $x = x_0$!
If (and when) you get $0/0$, try something else! (Factor $(x-a)$ out of top and bottom...)
If a function has something like $\sqrt{x}-\sqrt{a}$ in it, try multiplying (top and bottom) with $\sqrt{x}+\sqrt{a}$

(idea: $u = \sqrt{x}, v = \sqrt{a}$, then $x-a = u^2-v^2 = (u-v)(u+v)$.)

Sandwich Theorem: If $f(x) \leq g(x) \leq h(x)$, for all x near a (but not at a), and
$\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$.

One-sided limits:

Motivation: the Heaviside function



The Heaviside function has no limit at 0; it can't make up its mind whether to be 0 or 1. But if we just look to either side of 0, everything is fine; on the left, H(0) `wants' to be 0, while on the right, H(0) `wants' to be 1.

It's because these numbers are different that the limit as we approach 0 does not exist; but the `one-sided' limits DO exist.

Limit from the right: $\lim\limits_{x\to a^+} f(x) = L$ means $f(x)$ is close to L
when x is close to, and bigger than, a

Limit from the left: $\lim\limits_{x\to a^-} f(x) = M$ means $f(x)$ is close to M
when x is close to, and smaller than, a

$\lim\limits_{x\to a} f(x) = L$ then means $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = L$
(i.e., both one-sided limits exist, and are equal)

Limits at infinity / infinite limits:

∞ represents something bigger than any number we can think of.

$\lim\limits_{x\to\infty} f(x) = L$ means $f(x)$ is close of L when x is really large.

$\lim\limits_{x\to-\infty} f(x) = M$ means $f(x)$ is close of M when x is really large and *negative.*

Basic fact: $\lim\limits_{x\to\infty} \dfrac{1}{x} = \lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$

More complicated functions: divide by the highest power of x in the denomenator.

$f(x), g(x)$ polynomials, degree of $f = n$, degree of $g = m$

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0 \text{ if } n < m$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = (\text{coeff of highest power in } f)/(\text{coeff of highest power in } g) \text{ if } n = m$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty \text{ if } n > m$$

$\lim\limits_{x\to a} f(x) = \infty$ means $f(x)$ gets really large as x gets close to a
Also have $\lim\limits_{x\to a} f(x) = -\infty$; $\lim\limits_{x\to a^+} f(x) = \infty$; $\lim\limits_{x\to a^-} f(x) = \infty$; etc....

Typically, an infinite limit occurs where the denominator of $f(x)$ is zero
(although not always)

Asymptotes:

The line $y = a$ is a horizontal asymptote for a function f if
$\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$ is equal to a.
I.e., the graph of f gets really close to $y = a$ as $x \to \infty$ or $a \to -\infty$

The line $x = b$ is a vertical asymptote for f if $f \to \pm\infty$ as $x \to b$ from the right or left.
If numerator and denomenator of a rational function have no common roots, then vertical asymptotes = roots of denom.

Continuity:

A function f is <u>continuous</u> (cts) <u>at</u> <u>a</u> if $\lim\limits_{x\to a} f(x) = f(a)$

This means: (1) $\lim\limits_{x\to a} f(x)$ exists ; (2) $f(a)$ exists ; and
(3) they're equal.

Limit theorems say (sum, difference, product, quotient) of cts functions are cts.
Polynomials are continuous at every point;
rational functions are continuous except where denom=0.
Points where a function is not continuous are called <u>discontinuities</u>
Four flavors:
removable: both one-sided limits are the same
jump: one-sided limts exist, not the same
infinite: one or both one-sided limits is ∞ or $-\infty$
oscillating: one or both one-sided limits DNE

Intermediate Value Theorem:

If $f(x)$ is cts at every point in an interval $[a, b]$, and M is between $f(a)$ and $f(b)$, then there is (at least one) c between a and b so that $f(c) = M$.

Application: finding roots of polynomials

Tangent lines:

Slope of tangent line = limit of slopes of secant lines; at $(a, f(a)$:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Notation: call this limit $f'(a)$ = derivative of f at a

Different formulation: $h = x - a$, $x = a + h$

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \text{limit of } \textit{difference quotient}$$

If $y = f(x)$ = position at 'time' x, then difference quotient = average velocity;
limit = instantaneous velocity.

Derivatives

The derivative of a function:

derivative = limit of difference quotient (two flavors: $h \to 0$, $x \to a$)
If $f'(a)$ exists, we say f is <u>differentiable</u> at a
Fact: f differentiable (diff'ble) at a, then f cts at a

Using $h \to 0$ notation: replace a with x (= variable), get $f'(x)$ = <u>new</u> <u>function</u>

Or: $f'(x) = \lim_{z \to x} \dfrac{f(z) - f(x)}{z - x}$

$f'(x)$ = the derivative of f = function whose values are the slopes of the tangent lines to the graph of y=$f(x)$. Domain = every point where the limit exists

Notation:

$f'(x) = \dfrac{dy}{dx} = \dfrac{d}{dx}(f(x)) = \dfrac{df}{dx} = y' = D_x f = Df = (f(x))'$

Differentiation rules:

$\dfrac{d}{dx}$(constant) $= 0$ $\quad \dfrac{d}{dx}(x) = 1$

$(f(x)$+$g(x))' = (f(x))'$+ $(g(x))'$ $\quad (f(x)$-$g(x))'$= $(f(x))'$- $(g(x))'$

(c$f(x))'$= c$(f(x))'$

$(f(x)g(x))'$= $(f(x))'g(x)$+ $f(x)(g(x))'$ $\quad (\dfrac{f(x)}{g(x)})'$= $\dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

$(x^n)'$= nx^{n-1} , for n = ~~natural number~~ ~~integer~~ rational number

$e^x)'$= e^x $\quad (a^x)'$= $a^x \ln a$ [see below!]

[[$(1/g(x))'$= -$g'(x)/(g(x))^2$]]

Higher derivatives:

$f'(x)$ is 'just' a <u>function</u>, so we can take its derivative!

$(f'(x))'$= $f''(x)$ $\quad (= y'' = \dfrac{d^2y}{dx^2} = \dfrac{d^2f}{dx^2})$

= second derivative of f

Keep going! $f'''(x)$ = 3rd derivative, $f^{(n)}(x)$ = nth derivative

Rates of change

Physical interpretation:

$f(t)$= position at time t

$f'(t)$= rate of change of position = velocity

$f''(t)$= rate of change of velocity = acceleration

$|f'(t)|$ = speed

Basic principle: for object to change direction (velocity changes sign),

$f'(t)$= 0 somewhere (IVT!)

Examples:

Free-fall: object falling near earth; $s(t) = s_0 + v_0 t - \dfrac{g}{2}t^2$

$s_0 = s(0)$ = initial position; v_0 = initial velocity; g= acceleration due to gravity

Economics:

$C(x)$ = cost of making x objects; $R(x)$ = revenue from selling x objects;

$P = R - C$ = profit

$C'(x)$ = marginal cost = cost of making 'one more' object

$R'(x)$ = marginal revenue ; profit is maximized when $P'(x) = 0$;

i.e., $R'(x) = C'(x)$

Derivatives of trigonometric functions

Basic limit: $\lim_{x\to 0}\frac{\sin x}{x}=1$; everything else comes from this! $\lim_{h\to 0}\frac{1-\cos h}{h}=0$

Note: this uses radian measure!

$\lim_{x\to 0}\frac{\sin(bx)}{x}=\lim_{x\to 0}b\frac{\sin(bx)}{bx}=\lim_{u\to 0}b\frac{\sin(u)}{u}=b$

Then we get:

$(\sin x)'=\cos x$ $\quad$ $(\cos x)'=-\sin x$

$(\tan x)'=\sec^2 x$ $\quad$ $(\cot x)'=-\csc^2 x$

$(\sec x)'=\sec x\tan x$ $\quad$ $(\csc x)'=-\csc x\cot x$

The Chain Rule

Composition $(g\circ f)(x_0)=g(f(x_0))$; (note: we don't know what $g(x_0)$ is.)

$(g\circ f)'$ ought to have something to do with $g'(x)$ and $f'(x)$

in particular, $(g\circ f)'(x_0)$ should depend on $f'(x_0)$ and $g'(f(x_0))$

Chain Rule: $(g\circ f)'(x_0)=g'(f(x_0))f'(x_0)$

$=(d(\text{outside})\text{ eval'd at inside fcn})\cdot(d(\text{inside}))$

Ex: $((x^3+x-1)^4)'=(4(x^3+1-1)^3)(3x^2+1)$

Different notation:

$y=g(f(x))=g(u)$, where $u=f(x)$, then $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

Parametric equations: a general curve needn't be the graph of a function. But we can imagine ourselves travelling along a curve, and then $x=x(t)$ and $y=y(t)$ are functions of t=time. We still may have a reasonable tangent line to the graph, and its slope should still be

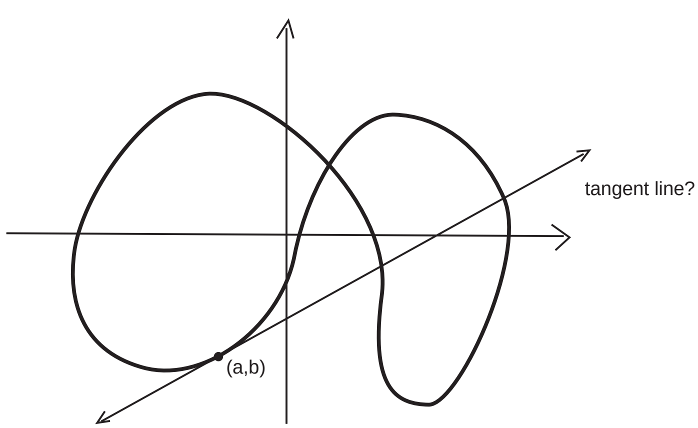
(change in y/change in $x=\lim_{t\to t_0}\frac{y(t)-y(t_0)}{x(t)-x(t_0)}=\lim_{t\to t_0}\frac{(y(t)-y(t_0))/(t-t_0)}{(x(t)-x(t_0))/(t-t_0)}$

$=\frac{\lim_{t\to t_0}(y(t)-y(t_0))/(t-t_0)}{\lim_{t\to t_0}(x(t)-x(t_0))/(t-t_0)}=\frac{y'(t_0)}{x'(t_0)}$

Implicit differentiation

We can differentiate functions; what about *equations*? (e.g., $x^2+y^2=1$)

graph looks like it has tangent lines



Idea: Pretend equation defines y as a function of x : $x^2 + (f(x))^2 = 1$ and differentiate!

$2x + 2f(x)f'(x) = 0$; so $\quad f'(x) = \dfrac{-x}{f(x)} = \dfrac{-x}{y}$

Different notation:

$x^2 + xy^2 - y^3 = 6$; then $2x + (y^2 + x(2y\dfrac{dy}{dx}) - 3y^2\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-2x - y^2}{2xy - 3y^2}$

Application: extend the power rule

$\dfrac{d}{dx}(x^r) = rx^{r-1}$ works for any *rational* number r

($y = x^{p/q}$ means $y^q = x^p$; differentiate!)

Inverse functions and their derivatives

Basic idea: run a function backwards

y=f(x) ; 'assign' the value x to the input y ; x=g(y)

need g a function; so need f is one-to-one

f is one-to-one: if f(x)=f(y) then x=y ; if $x \neq y$ then $f(x) \neq f(y)$

$g = f^{-1}$, then $g(f(x)) = x$ and $f(g(x)) = x$ (i.e., $g \circ f$=Id and $f \circ g$=Id)

finding inverses: rewrite y=f(x) as x=some expression in y

graphs: if (a,b) on graph of f, then (b,a) on graph of f^{-1}

graph of f^{-1} is graph of f, reflected across line y=x

horizontal lines go to vertical lines; horizontal line test for inverse

derivative of the inverse: $\quad f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$

if $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$

Logarithms

$f(x)$=a^x is **either** always increasing ($a > 1$) **or** always decreasing ($a < 1$)

inverse is $g(x) = \log_a x = \dfrac{\ln x}{\ln a}$

$\ln x$ is the inverse of e^x.

$\ln x$ is a log; it turns products into sums: $\ln(ab) = \ln(a) + \ln(b)$

$\ln(a^b) = b\ln(a)$; $\ln(a/b) = \ln(a) - \ln(b)$

$e^{\ln x} = x$ and $(e^x)' = e^x$, so $1 = (e^{\ln x})' = (e^{\ln x})(\ln x)' = x(\ln x)'$, so $(\ln x)' = 1/x$.

$\dfrac{d}{dx}(\ln x) = 1/\mathrm{x}$; $\dfrac{d}{dx}(\ln(f(x))) = \dfrac{f'(x)}{f(x)}$ This gives us:

Logarithmic differentiation: $f'(x) = f(x)\dfrac{d}{dx}(\ln(f(x)))$

useful for taking the derivative of products, powers, and quotients

$\ln(a^b)$ **should be** $b\ln a$, so $a^b = e^{b\ln a}$

$a^{b+c} = a^b a^c$; $a^{bc} = (a^b)^c$

$a^x = e^{x\ln a}$; $\dfrac{d}{dx}(a^x) = a^x \ln a$

$x^r = e^{r\ln x}$ (makes sense for *any real number* r) ; $\dfrac{d}{dx}(x^r) = e^{r\ln x}(r)(\dfrac{1}{x}) = rx^{r-1}$

Inverse trigonometric functions

Trig functions ($\sin x$, $\cos x$, $\tan x$, etc.) aren't one-to-one; make them!

$\sin x$, $-\pi/2 \le x \le \pi/2$ is one-to-one; inverse is Arcsin x

sin(Arcsin x)=x, all x; Arcsin($\sin x$)=x IF x in range above

$\tan x$, $-\pi/2 < x < \pi/2$ is one-to-one; inverse is Arctan x

tan(Arctan x)=x, all x; Arctan($\tan x$)=x IF x in range above

$\sec x$, $0 \le x < \pi/2$ and $\pi/2 < x \le \pi$, is one-to-one; inverse is Arcsec x

sec(Arcsec$x$)=x, all x; Arcsec($\sec x$)=x IF x in range above

Computing cos(Arcsin x), tan(Arcsec x), etc.; use right triangles

The other inverse trig functions aren't very useful,
they are essentially the negatives of the functions above.

Derivatives of inverse trig functions

They are the derivatives of inverse functions! Use right triangles to simplify.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{\sec^2(\arctan x)} = \frac{1}{x^2+1}$$
$$\frac{d}{dx}(\text{arcsec } x) = \frac{1}{\sec(\text{arcsec } x)\tan(\text{arcsec } x)} = \frac{1}{|x|\sqrt{x^2-1}}$$

Related Rates

Idea: If two (or more) quantities are related (a change in one value means a change in others), then their rates of change are related, too.

$xyz = 3$; pretend each is a function of t, and differentiate (implicitly).

General procedure:
Draw a picture, describing the situation; label things with variables.
Which variables, rates of change do you know, or want to know?
Find an equation relating the variables whose *rates of change* you know or want to know.
Differentiate!
Plug in the values that you know.

Linear approximation and differentials

Idea: The tangent line to a graph of a function makes a good approximation to the function, near the point of tangency.
Tangent line to $y = f(x)$ at $(x_0, f(x_0)$: $L(x) = f(x_0) + f'(x_0)(x - x_0)$
$f(x) \approx L(x)$ for x near x_0
Ex.: $\sqrt{27} \approx 5 + \frac{1}{2\cdot 5}(27-25)$, using $f(x) = \sqrt{x}$
$(1+x)^k \approx 1 + kx$, using x_0=0

$\Delta f = f(x_0 + \Delta x) - f(x_0)$, then $f(x_0 + \Delta x) \approx L(x_0 + \Delta x)$ translates to
$\Delta f \approx f'(x_0) \cdot \Delta x$
differential notation: $df = f'(x_0)dx$
So $\Delta f \approx df$, when $\delta x = dx$ is small

In fact, $\Delta f - df$ = (diffrnce quot $-f'(x_0))\Delta x$ = (small)$\cdot$(small) = really small, goes like $(\Delta x)^2$

Applications of Derivatives

Extreme Values

c is an (absolute) maximum for a function $f(x)$ if $f(c) \geq f(x)$ for every other x
d is an (absolute) minimum for a function $f(x)$ if $f(d) \leq f(x)$ for every other x
max or min = extremum

Extreme Value Theorem: If f is a continuous function defined on a closed interval $[a, b]$, then f actually *has* a max and a min.

Goal: figure out where they *are*!
c is a relative max (or min) if $f(c)$ is $\geq f(x)$ (or $\leq f(x)$) for every x *near* c. Rel max or min = rel extremum.
An absolute extremum is either a rel extremum or an endpoint of the interval.

c is a critical point if $f'(c) = 0$ or does not exist.
A rel extremum is a critical point.

So absolute extrema occur either at critical points *or* at the endpoints.
So to find the abs max or min of a function f on an interval $[a, b]$:
(1) Take derivative, find the critical points.
(2) Evaluate f at each critical point and endpoint.
(3) Biggest value is maximum value, smallest is minimum value.

The Mean Value Theorem

You can (almost) recreate a function by knowing its derivative

Mean Value Theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences:
Rolle's Theorem: $f(a) = f(b) = 0$; between two roots there is a critical point.
So: If a function has no critical points, it has at *most* one root!
A function with $f'(x)$=0 is constant.
Functions with the same derivative (on an interval) differ by a constant.

The First Derivative Test

f is *increasing* on an interval if $x > y$ implies $f(x) > f(y)$
f is *decreasing* on an interval if $x > y$ implies $f(x) < f(y)$
If $f'(x) > 0$ on an interval, then f is increasing
If $f'(x) < 0$ on an interval, then f is decreasing

Local max's / min's occur at critical points; how do you tell them apart?
Near a local max, f is increasing, then decreasing; $f'(x) > 0$ to the left of the critical point, and $f'(x) < 0$ to the right.

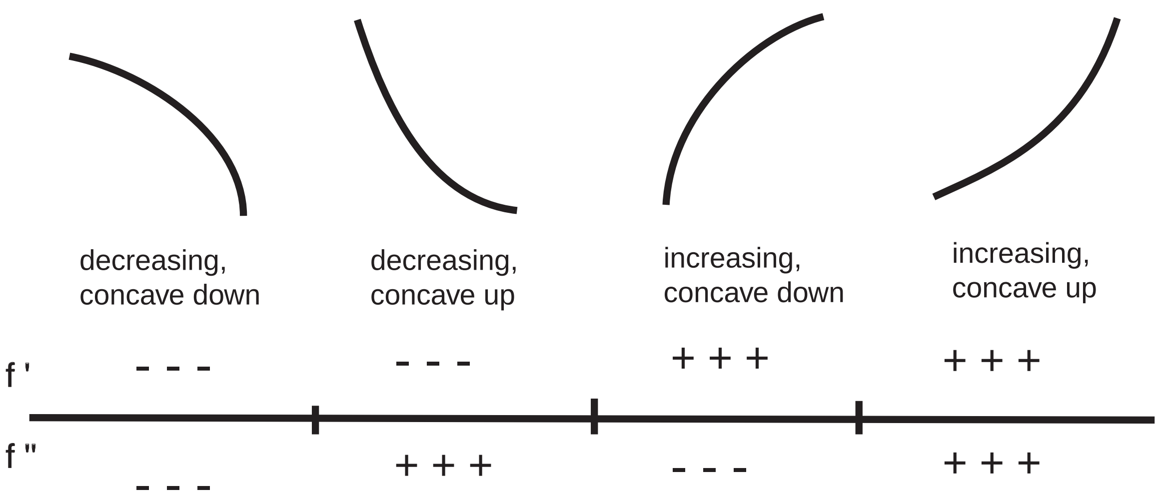
Near a local min, the opposite is true; $f'(x) < 0$ to the left of the critical point, and $f'(x) > 0$ to the right.
If the derivative does *not* change sign as you cross a critical point, then the critical point is not a rel extremum.
Basic use: plot where a function is increasing/decreasing: plot critical points; in between them, sign of derivative does not change.

The second derivative test and graphing

When we look at a graph, we see where function is increasing/decreasing. We also see:

f is concave up on an interval if $f''(x) > 0$ on the interval
Means: f' is increasing; f is *bending* up.
f is concave down on an interval if $f''(x) < 0$ on the interval
Means: f' is decreasing; f is *bending* down.
A point where the concavity changes is called a point of inflection

Graphing:
Find where $f'(x)$ and $f''(x)$ are 0 or DNE
Plot on the same line.
In between points, derivative and second derivative don't change sign, so graph looks like one of:



Then string together the pieces!
Use information about vertical and horizontal asymptotes to finish sketching the graph.

Second derivative test: If c is a critical point and
$f''(c) > 0$, then c is a rel min (smiling!)
$f''(c) < 0$, then c is a rel max (frowning!)

Newton's method:

A really fast way to approximate roots of a function.
Idea: tangent line to the graph of a function "points towards" a root of the function. But: roots of (tangent) lines are computationally straighforward to find!

$L(x) = f(x_0) + f'(x_0)(x - x_0)$; root is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Now use x_1 as starting point for new tangent line; keep repeating!

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Basic fact: if x_n approximates a root to k decimal places, then x_{n+1} tends to approximate it to $2k$ decimal places! BUT:

Newton's method might find the "wrong" root: Int Value Thm might find one, but N.M. finds a different one!
Newton's method might crash: if $f'(x_n) = 0$, then we can't find x_{n+1} (horizontal lines don't have roots!)
Newton's method might wander off to infinity, if f has a horizontal asymptote; an initial guess too far out the line will generate numbers even farther out.
Newton's method can't find what doesn't exist! If f has no roots, Newton's method will try to "find" the function's closest approach to the x-axis; but everytime it gets close, a nearly horizontal tangent line sends it zooming off again...

Optimization

This is really just finding the max or min of a function on an interval, with the added complication that you need to figure out *which* function, and *which* interval! Solution strategy is similar to a *related rates* problem:

Draw a picture; label things.
What do you need to maximize/minimize? Write down a formula for the quantity.
Use other information to eliminate variables, so your quantity depends on only one variable.
Determine the largest/smallest that the variable can reasonably be (i.e., find your interval)
Turn on the max/min machine!

L'Hôpital's Rule

indeterminate forms: limits which 'evaluate' to 0/0 ; e.g. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

LR# 1: If $f(a) = g(a) = 0$, f and g both differentiable near a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note: we can repeatedly apply L'Hôpital's rule to compute a limit, so long as the condition that top and bottom both tend to 0 holds for the new limit. Once this doesn't hold, L'Hôpital's rule can no longer be applied!

Other indeterminate forms: $\dfrac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

LR#2: if $f, g \to \infty$ as $x \to a$, then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Other cases: try to turn them into $0/0$ or ∞/∞. In the $0 \cdot \infty$ case, we can do this by throwing one factor or the other into the denominator (whichever is more tractable. In the last three cases, do this by taking logs, first.

Integration

Antiderivatives.

Integral calculus is all about finding areas of things, e.g. the area between the graph of a function f and the x-axis. This will, in the end, involve finding a function F whose *derivative* is f.

F is an *antiderivative* (or (indefinite) *integral*) of f if $F'(x) = f(x)$.
Notation: $F(x) = \int f(x)\ \mathrm{d}x$; it <u>means</u> $F'(x)=f(x)$; "the integral of f of x dee x"

Every differentiation formula we have encountered can be turned into an antidifferentiation formula; if g is the derivative of f, then f is an antiderivative of g. Two functions with the same derivative (on an interval) differ by a constant, so <u>all</u> antiderivatives of a function can be found by finding one of them, and then adding an arbitrary constant C.

Basic list:

$\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$) $\qquad \int 1/x\ \mathrm{d}x = \ln|x| + C$

$\int \sin(kx)\ \mathrm{d}x = \dfrac{-\cos(kx}{k} + C \qquad \int \cos(kx)\ \mathrm{d}x = \dfrac{\sin(kx)}{k} + C$

$\int \sec^2 x\ \mathrm{d}x = \tan x + C \qquad \int \csc^2 x\ \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x\ \mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

$\int e^x\ \mathrm{d}x = e^x + C$

Most differentiation rules can be turned into integration rules (although some are harder than others; some we will wait awhile to discover).

Basic integration rules: sum and constant multiple rules are straighforward to reverse: for k=constant,

$\int k \cdot f(x)\ \mathrm{d}x = k \int f(x)\ \mathrm{d}x \qquad \int (f(x) \pm g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$

Sums and Sigma Notation.

Idea: a lot of things can estimated by adding up alot of tiny pieces.

Sigma notation: $\displaystyle\sum_{i=1}^{n} a_i = a_1 + \cdots a_n$; just add the numbers up

Formal properties: $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Some things worth adding up:
length of a curve: approximate curve by a collection of straight line segments

length of curve $\approx \sum$(length of line segments)

distance travelled = (average velocity)(time of travel)
over short periods of time, avg. vel. $\approx$ instantaneous vel.
so distance travelled $\approx \sum$(inst. vel.)(short time intervals)

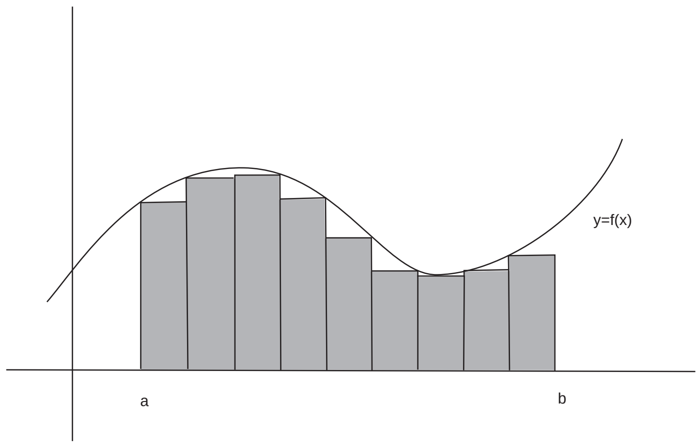
Average value of a function:

Average of n numbers: add the numbers, divide by n . For a function, add up lots of values of f, divide by number of values.

avg. value of $f \approx \frac{1}{n}\sum_{i=1}^{n} f(c_i)$

Area and Definite Integrals.

Probably the most important thing to approximate by sums: area under a curve.
Idea: approximate region b/w curve and x-axis by things whose areas we can easily calculate: **rectangles**!



Area between graph and x-axis $\approx \sum$ (areas of the rectangles) $=\sum_{i=1}^{n} f(c_i)\Delta x_i$

where c_i is chosen inside of the i-th interval that we cut $[a,b]$ up into. This is a Riemann sum for the function f on the interval $[a,b]$.)

We define the area to be the limit of these sums as the lengths of the subintervals gets small (so the number of rectangles goes to ∞, and call this the *definite integral* of f from a to b:

$$\int_a^b f(x)\ dx \ = \ \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$$

More precisely, we can at all Riemann sums, and look at what happens when the length Δx_i of the largest subinterval (call it Δ) gets small. If the Riemann sums all approximate some number I when Δ is small enough, then we call I the definite integral of f from a to b. But when do such limits exist?

Theorem If f is continuous on the interval $[a,b]$, then $\int_a^b f(x)\ dx$ exists.

(i.e., the area under the graph is approximated by rectangles.)

But this isn't how we want to compute these integrals! Limits of sums is very cumbersome. Instead, we try to be more systematic.

Properties of definite integrals:

First note: the sum used to define a definite integral doesn't need to have $f(x) \geq 0$; the limit still makes sense. When f is bigger than 0, we interpret the integral as area under the graph.

Basic properties of definite integrals:

$\int_a^a f(x)\ dx = 0$ $\qquad$ $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$ $\qquad$ $\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$

If $m \leq f(x) \leq M$ for all x in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$

More generally, if $f(x) \leq g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$

Average value of f : formalize our old idea! $\qquad \text{avg}(f) = \dfrac{1}{b-a}\int_a^b f(x)\ dx$

Mean Value Theorem for integrals: If f is continuous in $[a,b]$, then there is a c in $[a,b]$ so that $f(c) = \dfrac{1}{b-a}\int_a^b f(x)\ dx$

The fundamental theorems of calculus.

Formally, $\int_a^b f(x)\ dx$ depends on a and b. Make this explicit:

$\int_a^x f(t)\ dt = F(x)$ is a function of x.

$F(x) =$ the area under the graph of f, from a to x.

Fund. Thm. of Calc (# 1): If f is continuous, then $F'(x) = f(x)$ $\quad$ (F is an antiderivative of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \int_a^b f(t)\ dt$, we get

Fund. Thm. of Calc (# 2): If f is continuous, and F is an antiderivative of f, then $\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\ |_a^b$

Ex: $\int_0^\pi \sin x\ dx = (-\cos\pi) - (-\cos 0) = 2$

FTC # 2 makes finding antiderivatives very important! FTC # 1 gives a method for building antiderivatives:

$F(x) = \int_a^x \sqrt{\sin t}\ dt$ is an antiderivative of $f(x) = \sqrt{\sin x}$

$G(x) = \int_{x^2}^{x^3} \sqrt{1+t^2}\ dt = F(x^3) - F(x^2)$, where

$F'(x) = \sqrt{1+x^2}$, so $G'(x) = F'(x^3)(3x^2) - F'(x^2)(2x)$...