1. Find the following integrals (10 pts. each):

(a): $\int_1^4 x^2 \ln x \, dx$ by parts! $u = \ln x$ $dv = x^2 dx$

$du = \frac{1}{x} dx$ $v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \ln|x| \Big|_1^4 - \int_1^4 \frac{1}{3}x^2 dx = \frac{1}{3}x^3 \ln|x| \Big|_1^4 - \frac{1}{9}x^3 \Big|_1^4$

$= \left(\frac{1}{3}4^3 \ln(4) - \frac{1}{3}1^3 \ln(1)\right) - \frac{1}{9}(4^3 - 1^3)$

[can also be done by $u = x^2 \ln x$, $dv = dx$!]

(b): $\int \sin^2 x \cos^3 x \, dx$ $= \int \sin^2 x \cos^2 x \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$ $u = \sin x$ $du = \cos x \, dx$

$= \int u^2 (1 - u^2) \, du \Big|_{u = \sin x} = \int u^2 - u^4 du \Big|_{u = \sin x}$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c \Big|_{u = \sin x} = \frac{1}{3}(\sin x)^3 - \frac{1}{5}(\sin x)^5 + C$

2. When you apply the appropriate trigonometric substitutions, what do the following integrals become?

(a): $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

$x = 2\sin u \quad dx = 2\cos u\,du$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2 u} = 2\cos u$

$$= \boxed{\int \frac{(2\cos u)(2\cos u\,du)}{(2\sin u)^2}}\Bigg|_{x=2\sin u} = \int \frac{\cos^2 u}{\sin^2 u}\,du\Bigg|_{x=2\sin u}$$

(b): $\int \frac{x^2}{\sqrt{4x^2+9}}\,dx$

want $\quad 4x^2+9 = 9\tan^2 u + 9 = 9\sec^2 u$

$2x = 3\tan u \quad x = \frac{3}{2}\tan u$

$dx = \frac{3}{2}\sec^2 u\,du$

$\sqrt{4x^2+9} = \sqrt{9\sec^2 u} = 3\sec u$

$$= \boxed{\int \frac{(\frac{3}{2}\tan u)^2 (\frac{3}{2}\sec^2 u\,du)}{3\sec u}}\Bigg|_{x=\frac{3}{2}\tan u} = \int \frac{3}{4}\tan^2 u \sec u\,du\Bigg|_{x=\frac{3}{2}\tan u}$$

3. Use a comparison theorem to determine whether or not the following improper integral converges:

$$\int_2^\infty \frac{\sqrt{1+x+x^3}}{x^2-1}\,dx = \int_2^\infty f(x)\,dx$$

$\sqrt{1+x+x^3} \approx \sqrt{x^3}$ for x large

$x^2-1 \approx x^2$ for x large

$$\frac{\frac{\sqrt{1+x+x^3}}{x^2-1}}{\frac{\sqrt{x^3}}{x^2}} = \sqrt{\frac{1+x+x^3}{x^3}} \cdot \frac{x^2}{x^2-1} \to \sqrt{1}\cdot 1 = 1 \neq 0$$

as $x \to \infty$

since $\frac{1+x+x^3}{x^3} = \frac{\frac{1}{x^3}+\frac{1}{x^2}+1}{1} \to \frac{0+0+1}{1} = 1$

as $x \to \infty$

and $\frac{x^2}{x^2-1} = \frac{1}{1-\frac{1}{x^2}} \to \frac{1}{1-0} = 1$

So since $\int_2^\infty \frac{\sqrt{x^3}}{x^2}\,dx = \int_2^\infty \frac{x^{3/2}}{x^2}\,dx = \int_2^\infty \frac{dx}{x^{1/2}}$

diverges $(\frac{1}{2} < 1)$, $\int_2^\infty f(x)\,dx$ <u>diverges</u>

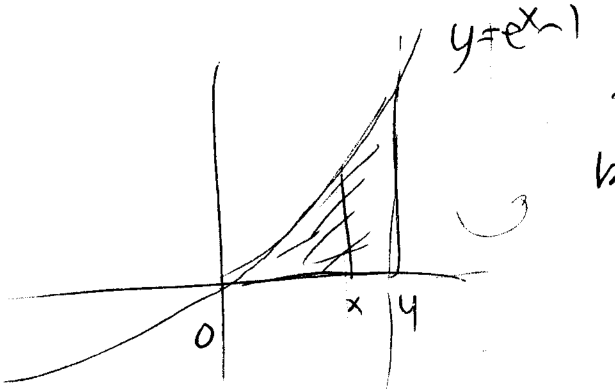
[can also be done by <u>direct</u> comparison with

$\frac{\sqrt{x^3}}{x^2} \leq \frac{\sqrt{1+x+x^3}}{x^2-1}$, so top is smaller and bottom is bigger.]

4. Let R be the region in the plane lying between the graph of the function $y = e^x - 1$, the vertical line $x = 4$ and the x-axis. Set up, **but do not evaluate**, the definite integrals which compute the volumes of the solids of revolution obtained by revolving the region R around

(a) the vertical line $x = 6$ (b) the x-axis

Be sure to indicate which is which!



top $= e^x - 1$ left $= 0$
bottom $= 0$ right $= 4$

vertical line : $\rightarrow$ shells

$$\text{Volume} = \int_{\text{left}}^{\text{right}} 2\pi(\text{radius})(\text{height})\,dx$$

$$= \boxed{\int_0^4 2\pi(6-x)(e^x-1-0)\,dx}$$

horizontal line $\rightarrow$ slices $\rightarrow$ slice = disk

$$\text{Volume} = \int_{\text{left}}^{\text{right}} \pi(\text{radius})^2\,dx$$

$$= \boxed{\int_0^4 \pi(e^x-1)^2\,dx}$$

[can also be done dy : right $= 4$, left $= \ln(y+1)$, from bottom $= 0$ to top $= e^4 - 1$.]

5. Set up, **but do not evaluate**, the definite integral which computes the arclength of the graph of the function $y = \ln x$ between $x = 5$ and $x = 11$.

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$$\text{Length} = \boxed{\int_5^{11} \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx}$$

6. (15 pts.) Recall that if a function f has second derivative satisfying $|f''(x)| \leq M$ for every x in the interval $[a, b]$, then the error E_n in approximating the integral $\int_a^b f(x)\,dx$ using the trapezoidal rule using n equal subintervals is at most

$$M\frac{(b-a)^3}{12n^2}$$

Based on this, how many subintervals should we divide the interval $[2, 5]$ into in order to be sure to approximate the integral $\int_2^5 x \ln x \, dx$ with an error of less than $\frac{1}{100}$?

$f(x) = x \ln x$

$f'(x) = 1 \cdot \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$

$f''(x) = \frac{1}{x} + 0 = \frac{1}{x} \leftarrow$ decreasing! $\left(f'''(x) = -\frac{1}{x^2} < 0\right)$

so max is at $x = 2$ $\quad \underline{M = \frac{1}{2}}$

want error $< \frac{1}{100}$, error $\leq \frac{1}{2} \frac{(5-2)^3}{12n^2}$

so want $\frac{1}{2} \frac{3^3}{12n^2} < \frac{1}{100}$ so $n^2 > 100 \frac{1}{2} \frac{3^3}{12}$

$= 50 \frac{27}{12} = 50 \frac{9}{4}$

$= \frac{450}{4} = 112.5$

so need $n > \sqrt{112.5} \approx 10.6$

so $n = 11$ will work.