Solutions

1. (20 pts.) Cesium-137, denoted Cs_{137}, is a radioactive substance with a half-life of 30 years. That is, if $C(t)$ represents the amount of Cs_{137} in a sample after t years, then

$$C(30) = \frac{1}{2}C(0) .$$

If we start with a 4 gram sample of Cs_{137}, how much Cs_{137} will remain after 10 years?

$$C(t) = C(0)e^{kt} = 4e^{kt}$$

$$C(30) = \frac{1}{2}C(0) = 2 = 4e^{30k}$$

$$\Rightarrow \frac{1}{2} = e^{30k} \quad \ln(\tfrac{1}{2}) = 30k \quad k = \frac{1}{30}\ln(\tfrac{1}{2})$$

$$\text{so} \quad C(t) = 4e^{\ln(\frac{1}{2})\frac{t}{30}}, \quad \text{so}$$

$$C(10) = \boxed{4e^{\ln(\frac{1}{2})\frac{10}{30}}}$$

$$= 4e^{\ln((\frac{1}{2})^{1/3})} = 4(\tfrac{1}{2})^{1/3}$$

2. (10 pts. each) Determine whether each of the following series converges or diverges:

(a): $\sum_{n=0}^{\infty}(-1)^n \frac{n}{n^3+1}$ $= \sum a_n$

$|a_n| = \frac{n}{n^3+1}$ limit compare with $\frac{n}{n^3} = \frac{1}{n^2}$

$\frac{|a_n|}{1/n^2} = \frac{n^3}{n^3+1} = \frac{1}{1+\frac{1}{n^3}} \to \frac{1}{1+0} = 1 \neq 0$ so

since $\sum \frac{1}{n^2}$ converges (p-series $p=2>1$), $\sum |a_n|$

converges, so $\sum a_n$ $\boxed{\text{converges}}$!

(b): $\sum_{n=0}^{\infty} \frac{2n}{3n+5}$ $= \sum a_n$

$\frac{2n}{3n+5} = \frac{2}{3+5/n} \to \frac{2}{3+0} = \frac{2}{3} \neq 0$ as $n \to \infty$,

so $\sum a_n$ $\boxed{\text{diverges}}$ by the n^{th} term test.

3. (10 pts. each) Find the radius of convergence of each of the following power series:

(a): $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n}$ $= \sum a_n x^n$ $\quad a_n = \frac{1}{n\sqrt{n}\,3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)(n+1)^{1/2}3^{n+1}}}{\frac{1}{n\sqrt{n}\,3^n}} = \frac{n\sqrt{n}\,3^n}{(n+1)\sqrt{n+1}\,3^{n+1}} = \frac{1}{3}\frac{n}{n+1}\sqrt{\frac{n}{n+1}}$$

$$\to \frac{1}{3}(1)(1) = \frac{1}{3} = L \text{ as } n\to\infty, \text{ so}$$

R = radius of convergence $= \frac{1}{L} = \boxed{3}$

(b) $\sum_{n=0}^{\infty} \frac{2^n}{n!}x^n$ $= \sum a_n x^n$ $\quad a_n = \frac{2^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \frac{2}{n+1} \to 0$$

as $n\to\infty$ so $R = \text{"}\frac{1}{L}\text{"} = \text{"}\frac{1}{0}\text{"} = \boxed{\infty}$ = radius of conv.

4. (20 pts.) Using the Taylor series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, find a power series representation for the function

$$f(x) = \frac{x^2}{1+x^4}$$

centered at $x = 0$ (by an appropriate substitution and multiplication). Use this to find a series which converges to the integral

$$\int_0^{1/3} \frac{x^2}{1+x^4}\, dx\ .$$

$$f(x) = \frac{x^2}{1+x^4} = x^2 \frac{1}{1+x^4} = x^2 \frac{1}{1-(-x^4)}$$

$$= x^2 \sum_{n=0}^{\infty} (-x^4)^n = x^2 \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{4n+2}$$

So $\int \frac{x^2}{1+x^4} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{4n+3} = g(x) \ (+C)$

So $\int_0^{1/3} \frac{x^2}{1+x^4} dx = g(\tfrac{1}{3}) - g(0)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3}\left(\frac{1}{3}\right)^{4n+3} - \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3}(0)^{4n+3}\right) = 0 \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3}\left(\frac{1}{3}\right)^{4n+3}}$$

according to Maple 13.....

FYI: $\int \frac{x^2}{1+x^4} dx = -\frac{\sqrt{2}}{8} \ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x+1) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x-1) + C$

5. (20 pts.) Find the Taylor polynomial of degree 3, centered at $x = 2$, for the function

$$f(x) = x^6 + x + 5 \ .$$

$f(2) = 2^6 + 2 + 5 = 64 + 7 = 71$

$f'(\cancel{x}) = 6x^5 + 1$

$f'(2) = 6 \cdot 2^5 + 1 = 6 \cdot 32 + 1 = 192 + 1 = 193$

$f''(x) = 30x^4$

$f''(2) = 30 \cdot 2^4 = 30 \cdot 16 = 480$

$f'''(x) = 120x^3$

$f'''(2) = 120 \cdot 2^3 = 120 \cdot 8 = \cancel{480}\ 960$

So $P_3(x) = f(2) + f'(2)(x-2) + \dfrac{f''(2)}{2!}(x-2)^2 + \dfrac{f'''(2)}{3!}(x-2)^3$

$= \boxed{71 + 193(x-2) + \frac{480}{2}(x-2)^2 + \frac{960}{6}(x-2)^3}$

$= 71 + 193(x-2) + 240(x-2)^2 + 160(x-2)^3$