Solutions

1. ~~3~~. For $P = (-1, 2, 3)$ and $Q = (3, 1, 2)$, set $\vec{v} = \overrightarrow{PQ}$.

(a) (10 pts.) Express the vector $\vec{v}$ in component form (i.e., as $< a, b, c >$).

$$\overrightarrow{PQ} = \langle 3-(-1), 1-2, 2-3 \rangle = \langle 4, -1, -1 \rangle$$

(b) (10 pts.) Find a vector which points in the same direction as $\vec{v}$ but has length 8. [You may express your answer in either $\overrightarrow{RS}$ form or component form.]

$$\|\overrightarrow{PQ}\| = \left(4^2 + (-1)^2 + (-1)^2\right)^{1/2} = (16+1+1)^{1/2} = \sqrt{18}$$

~~8~~ $\frac{8}{\sqrt{18}} \langle 4, -1, -1 \rangle$ has length 8 and same direction

2 ~~1~~. For the vectors $\vec{v} =< 1, 2, 2 >$ and $\vec{w} =< 1, -1, 2 >$, find

(a) (10 pts.) the cosine of the angle between $\vec{v}$ and $\vec{w}$, and

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{(1)(1)+(2)(-1)+(2)(2)}{(1^2+2^2+2^2)^{1/2}\,(1^2+(-1)^2+2^2)^{1/2}}$$

$$= \boxed{\frac{3}{\sqrt{9}\sqrt{6}}} = \frac{3}{3\sqrt{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

(b) (10 pts.) the orthogonal projection $\mathrm{proj}_{\vec{w}}(\vec{v})$ of the vector $\vec{v}$ in the direction of the vector $\vec{w}$.

$$\mathrm{proj}_w(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{3}{1^2+(-1)^2+2^2}\langle 1,-1,2\rangle$$

$$= \boxed{\frac{3}{6}\langle 1,-1,2\rangle} = \left\langle \frac{1}{2}, \frac{-1}{2}, 1\right\rangle$$

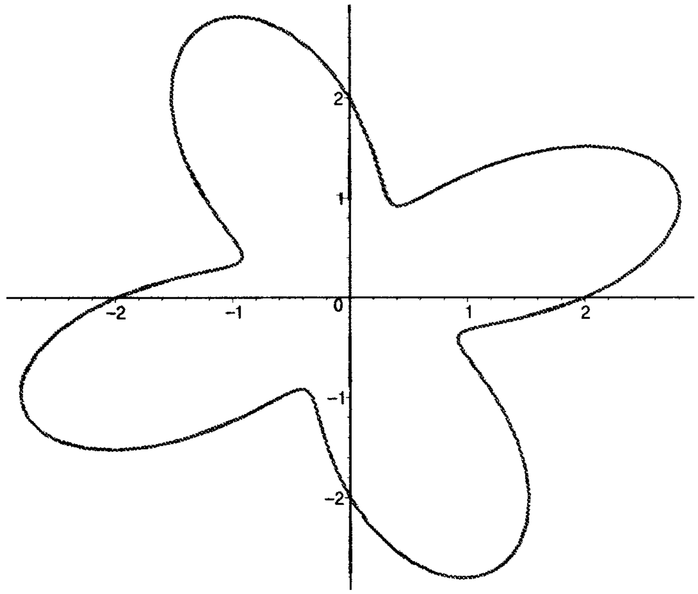
$$\boxed{\begin{aligned} \text{Check!}\quad & \vec{v} - \mathrm{proj}_{\vec{w}}(\vec{v}) = \langle 1,2,2\rangle - \left\langle \tfrac{1}{2}, \tfrac{-1}{2}, 1\right\rangle \\ \vec{u}\quad & = \left\langle \tfrac{1}{2}, \tfrac{5}{2}, 1\right\rangle \\ & \vec{u}\cdot\vec{w} = \tfrac{1}{2} - \tfrac{5}{2} + 2 = \tfrac{5}{2} - \tfrac{5}{2} = 0 \checkmark \end{aligned}}$$

Solutions

3 ~~4~~. (20 pts.) Find the (rectangular!) **equation for the line** tangent to the polar curve

$$r = 2 + \sin(4\theta)$$

when $\theta = 0$.



$x = r\cos\theta$
$= 2\cos\theta + \sin(4\theta)\cos\theta$

$y = r\sin\theta = 2\sin\theta + \sin(4\theta)\sin(\theta)$

$\frac{dx}{d\theta} = -2\sin\theta + 4\cos(4\theta)\cos\theta + \sin(4\theta)(-\sin\theta)$

$\frac{dy}{d\theta} = 2\cos\theta + 4\cos(4\theta)\sin\theta + \sin(4\theta)\cos\theta$

At $\theta = 0$:
$\frac{dx}{d\theta} = -2(0) + 4(1)(1) + (0)(-0) = 0 + 4 + 0 = 4$
$\frac{dy}{d\theta} = 2(1) + 4(1)(0) + (0)(1) = 2 + 0 + 0 = 2$

So $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2}{4} = \frac{1}{2}$

At $\theta = 0$
$x = 2(1) + (0)(1) = 2$
$y = 2(0) + (0)(0) = 0$

So tangent line is $\boxed{y - 0 = \frac{1}{2}(x - 2)}$

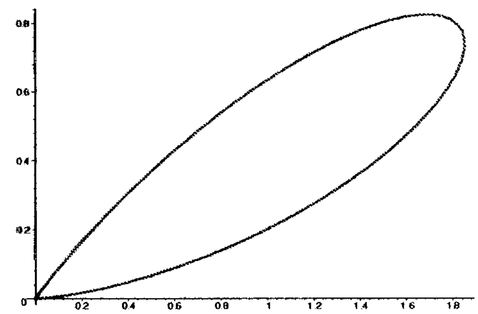
$y = \frac{1}{2}x - 1$

4. (20 pts.) Find the area enclosed by the graph of the polar curve

$$r = 1 - \cos(8\theta) \quad = f(\theta)$$

between $\theta = 0$ and $\theta = \dfrac{\pi}{4}$.

[Hint: Recall that $\cos(A)\cos(B) = \dfrac{1}{2}[\cos(A+B) + \cos(A-B)]$.]



$$\text{Area} = \int_{\text{start}}^{\text{finish}} \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1-\cos(8\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - 2\cos(8\theta) + \cos^2(8\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} 1 - 2\cos(8\theta) + \frac{1}{2}\left(\cos(16\theta) + \cos(0)\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{3}{2} - 2\cos(8\theta) + \frac{1}{2}\cos(16\theta)\, d\theta$$

$$= \frac{1}{2}\left(\frac{3}{2}\theta - \frac{2}{8}\sin(8\theta) + \frac{1}{32}\sin(16\theta)\right)\Bigg|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\left(\frac{3}{2}\frac{\pi}{4} - \frac{2}{8}\sin(2\pi) + \frac{1}{32}\sin(4\pi)\right) - \left(0 - \frac{2}{8}\sin(0) + \frac{1}{32}\sin(0)\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{3\pi}{8} - 0 + 0\right) - (0 - 0 + 0)\right) = \boxed{\frac{3\pi}{16}}$$

5. A spot on the edge of a spinning disk tossed through the air traces out a curve given by the vector-valued function 0;

$$\vec{r}(t) = (5t + \cos t, 10t - t^2 + \sin t) \ .$$

(a) **(15 pts.) Find the velocity and acceleration vectors for the spot at each time t.**

velocity $= \vec{r}\,'(t) = \langle 5 - \sin t, 0, 10 - 2t + \cos t \rangle$

acceleration $= \vec{r}\,''(t) = \langle -\cos t, 0, -2 - \sin t \rangle$

(b) (5 pts.) For what value(s) of t between 0 and ~~4π~~ 2π is the magnitude (i.e., length) of the acceleration vector the largest?

$$\|\vec{r}\,''(t)\| = \left((-\cos t)^2 + 0^2 + (-2 - \sin t)^2 \right)^{1/2}$$

$$= \left(\cos^2 t + 4 + 4\sin t + \sin^2 t \right)^{1/2}$$

$$= \left(5 + 4\sin t \right)^{1/2}$$

this is largest when $5 + 4\sin t$ is largest

$f(t)$

$f'(t) = 4\cos t = 0 \quad t = \frac{\pi}{2}, \frac{3\pi}{2}, \cancel{\frac{5\pi}{2}}, \cancel{\frac{7\pi}{2}}$

(also check endpts! $t=0$, $t = 2\pi$

$f(0) = 5 + 4\sin(0) = 5 \qquad f(\frac{\pi}{2}) = 4 + 5\sin(\frac{\pi}{2}) = 9$

$f(\frac{3\pi}{2}) = 5 + 4\sin(\frac{3\pi}{2}) = 5 - 4 = 1 \qquad f(2\pi) = 4 + 5\sin(2\pi) = 4$

largest when $\boxed{t = \frac{\pi}{2}}$