Math 107 Sections 151-155

Topics for the first exam

Integration

Basic list:

$\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$) $\qquad \int 1/x\ \mathrm{d}x = \ln|x| + C$

$\int \sin(kx)\ \mathrm{d}x = \frac{-\cos(kx)}{k} + C \qquad \int \cos(kx)\ \mathrm{d}x = \frac{\sin(kx)}{k} + C$

$\int \sec^2 x\ \mathrm{d}x = \tan x + C \qquad \int \csc^2 x\ \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x\ \mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

$\int \tan x\ \mathrm{d}x = \ln|\sec x| + C \qquad \int \sec x\ \mathrm{d}x = \ln|\sec x + \tan x| + C$

$\int \cot x\ \mathrm{d}x = \ln|\sin x| + C \qquad \int \csc x\ \mathrm{d}x = -\ln|\csc x + \cot x| + C$

$\int e^x\ \mathrm{d}x = e^x + C \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \text{Arcsin}(\frac{x}{a}) + c$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\ \text{Arctan}(\frac{x}{a}) + c \qquad \int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\text{Arcsec}(\frac{x}{a}) + c$

Basic integration rules: for k=constant,

$\int k \cdot f(x)\ \mathrm{d}x = k \int f(x)\ \mathrm{d}x \qquad \int (f(x) \pm g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$

The Fundamental Theorem of Calculus

$\int_a^x f(t)\ dt = F(x)$ is a function of x. $\quad F(x) =$ the area under graph of f, from a to x.

FTC 2: If f is cts, then $F'(x) = f(x) \quad$ (F is an antideriv of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \int_a^b f(t)\ dt$, we get

FTC 1: If f is cts, and F is an antideriv of f, then $\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\ |_a^b$

Integration by substitution. The idea: reverse the chain rule!

$g(x) = u$, then $\dfrac{d}{dx}f(g(x)) = \dfrac{d}{dx}f(u) = f'(u)\ \dfrac{du}{dx}$, so $\int f'(u)\ \dfrac{du}{dx}\ dx = \int f'(u)\ du = f(u) + c$

$\int f(g(x))g'(x)\ dx$; set $u = g(x)$, then $du = g'(x)\ dx$,

so $\int f(g(x))g'(x)\ dx = \int f(u)\ du$, <u>where</u> $u = g(x)$

Example: $\int x(x^2-3)^4\ dx$; set $u = x^2 - 3$, so du=$2x\ dx$. Then

$\int x(x^2-3)^4\ dx = \frac{1}{2}\int (x^2-3)^4 2x\ dx = \frac{1}{2}\int u^4\ du\ |_{u=x^2-3} = \frac{1}{2}\frac{u^5}{5} + c\ |_{u=x^2-3} = \frac{(x^2-3)^5}{10} + c$

The three most important points:

1. Make sure that you calculate (and then set aside) your du before doing step 2!
2. Make sure everything gets changed from x's to u's
3. **<u>Don't</u>** push x's through the integral sign! They're <u>not</u> constants!

We can use u-substitution directly with a definite integral, provided we remember that

$\displaystyle\int_a^b f(x)\ dx$ <u>really</u> means $\displaystyle\int_{x=a}^{x=b} f(x)\ dx$, and we remember to change <u>all</u> x's to u's!

Ex: $\displaystyle\int_1^2 x(1+x^2)^6\ dx$; set $u = 1+x^2$, $du = 2x\ dx$. when $x = 1$, $u = 2$; when $x = 2$, $u = 5$;

so $\displaystyle\int_1^2 x(1+x^2)^6\ dx = \frac{1}{2}\int_2^5 u^6\ du = ...$

Integration by parts

Product rule: $\mathrm{d}(uv) = (\mathrm{d}u)v + u(\mathrm{d}v)$

reverse: $\int u \; \mathrm{d}v = uv - \int v \; \mathrm{d}u$

Ex: $\int x \cos x \; \mathrm{d}x$: set $u{=}x$, $\mathrm{d}v{=}\cos x \; \mathrm{d}x$ $\quad$ $\mathrm{d}u{=}\mathrm{d}x$, $v = \sin x$ (or any <u>other</u> antiderivative)

So: $\int x \cos x = x \sin x - \int \sin x \; \mathrm{d}x = \ldots$

special case: $\displaystyle\int f(x) \; \mathrm{d}x$; $u = f(x)$, $\mathrm{d}v{=}\mathrm{d}x$ $\quad$ $\displaystyle\int f(x) \; \mathrm{d}x = xf(x) - \int xf'(x) \; \mathrm{d}x$

Ex: $\displaystyle\int \text{Arcsin } x \; \mathrm{d}x = x \text{ Arcsin } x - \int \frac{x}{\sqrt{1-x^2}} = \ldots$

The basic idea: integrate part of the function (a part that you <u>can</u>), differentiate the rest. Goal: reach an integral that is "nicer".

Ex: $\int x^3 \ln x \; dx = (x^4/4) \ln x - \int (x^4/4)(1/x) \; dx = \ldots$

Trig substitution

Idea: get rid of square roots, by turning the stuff inside into a perfect square!

$\sqrt{a^2 - x^2}$: set $x = a \sin u$. $\mathrm{d}x = a \cos u \; \mathrm{d}u$, $\sqrt{a^2 - x^2} = a \cos u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}} \; \mathrm{d}x = \int \frac{\cos u}{\sin^2 u \cos u} \; \mathrm{d}u\Big|_{x=\sin u} = \ldots$

$\sqrt{a^2 + x^2}$: set $x = a \tan u$. $\mathrm{d}x = a \sec^2 u \; \mathrm{d}u$, $\sqrt{a^2 + x^2} = a \sec u$

Ex: $\displaystyle\int \frac{1}{(x^2+4)^{3/2}} \; \mathrm{d}x = \int \frac{2\sec^2 u}{(2 \sec u)^3} \; \mathrm{d}u\Big|_{x=2\tan u} = \ldots$

$\sqrt{x^2 - a^2}$: set $x = a \sec u$. $\mathrm{d}x = a \sec u \tan u \; \mathrm{d}u$, $\sqrt{x^2 - a^2} = a \tan u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}} \; \mathrm{d}x = \int \frac{\sec u \tan u}{\sec^2 u \tan u} \; \mathrm{d}u\Big|_{x=\sec u} = \ldots$

Undoing the "u-substitution": use right triangles! (<u>Draw</u> a right triangle!)

Ex: $x = a \sin u$, then angle u has opposite $= x$, hypotenuse $= a$, so adjacent $= \sqrt{a^2 - x^2}$. So $\cos u = (\sqrt{a^2 - x^2})/a$, $\tan u = x/\sqrt{a^2 - x^2}$, etc.

Trig integrals: What trig substitution usually leads to!

$\displaystyle\int \sin^n x \; \cos^m x \; dx$

If n is odd, keep one $\sin x$ and turn the others, in pairs, into $\cos x$
(using $\sin^2 x = 1 - \cos^2 x$), then do a u-substitution $u = \cos x$.

If m is odd, reverse the roles of $\sin x$ and $\cos x$.

If both are even, turn the $\sin x$ into $\cos x$ (in pairs) and use the double angle formula

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

This will convert $\cos^m x$ into a bunch of *lower powers* of $\cos(2x)$;
odd powers can be dealt with by substitution, even powers by another application of the angle doubling formula!

$\displaystyle\int \sec^n x \; \tan^m x \; dx = \int \frac{\sin^m x}{\cos^{n+m} x} \; dx$

If n is *even*, set two of them aside and convert the rest to $\tan x$

using $\sec^2 x = \tan^2 x + 1$, and use $u = \tan x$.

If m is *odd*, set one each of $\sec x$, $\tan x$ aside, convert the rest of the $\tan x$ to $\sec x$ using $\tan^2 x = \sec^2 x - 1$, and use $u = \sec x$.

If n is odd and m is even, convert all of the $\tan x$ to $\sec x$ (in pairs), leaving a bunch of powers of $\sec x$. Then use the *reduction formula*:

$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$

At the end, reach $\int sec^2 x\ dx = \tan x + C$ or $\int \sec x\ dx = \ln|\sec x + \tan x| + C$

A little "trick" worth knowing:

the substitution $u = \frac{\pi}{2} - x$, since $\sin(\frac{\pi}{2} - x) = \cos x$ and $\cos(\frac{\pi}{2} - x) = \sin x$, will *reverse* the roles of $\sin x$ and $\cos x$,

so will turn $\cot x$ into $\tan u$ and $\csc x$ into $\sec u$. So, for example, the integral

$\int \frac{\cos^4 x}{\sin^7 x}\ dx = \int \csc^3 x \cot^4 x\ dx$, which our techniques don't cover,

becomes $\int \sec^3 u \tan^4 u\ du$, which our techniques do cover.

Partial fractions

rational function = quotient of polynomials

Idea: integrate by writing function as sum of simpler functions

Procedure: $f(x) = p(x)/q(x)$

(0): arrange for degree(p)<degree(q); do long division if it isn't

(1): factor $q(x)$ into linear and irreducible quadratic factors

(2): group common factors together as powers

(3a): for each group $(x-a)^n$ add together: $\frac{a_1}{x-a} + \cdots + \frac{a_n}{(x-a)^n}$

(3b): for each group $(ax^2+bx+c)^n$ add together:

$$\frac{a_1x+b_1}{ax^2+bx+c} + \cdots + \frac{a_nx+b_n}{(ax^2+bx+c)^n}$$

(4) set $f(x)$ = sum of all sums; solve for the 'undetermined' coefficients

put sum over a common denomenator (=$q(x)$); set numerators equal.

always works: multiply out, group common powers, set coeffs of the two polys equal

Ex: $x + 3 = a(x-1) + b(x-2) = (a+b)x + (-a-2b)$; $1 = a + b$, $3 = -a - 2b$

linear term $(x-a)^n$: set $x = a$, will allow you to solve for a coefficient

if $n \geq 2$, take derivatives of both sides! set x=a, gives another coeff.

$$\text{Ex: } \frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+1)} = \ldots$$

Numerical Integration

Sometimes (most times?) the Fundamental Theorem of Calculus won't help us to compute a definite integral; we can't find an antiderivative. So we need to fall back on the definition:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i \text{ approximates } \int_a^b f(x)\ dx,$$

where the interval $[a, b]$ is cut into n pieces of length $\Delta x_1, \ldots \Delta x_n$, and c_i lies in the i-th subinterval

Typically, for convenience, we choose the subintervals to have the same length $\Delta x_i = \Delta x = \frac{b-a}{n}$, and make "standard" choices of elements in the i-th subinterval $[x_{i-1}, x_i]$:

$L(f, n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x$ (left endpoint estimate)

$R(f, n) = \sum_{i=1}^{n} f(x_i)\Delta x$ (right endpoint estimate)

$M(f, n) = \sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2})\Delta x$ (midpoint estimate)

In the end though, each of these is throwing out a lot of information, since it approximates f on an interval by a constant. We can do better, taking into account more infmation about the function f, by approximating f by functions that better "fit" f on a subinterval, whose integrals we know how to compute.

We focus on linear functions: we replace f on each subinterval by the linear function having the same values at the endpoints. This essentially replaces a rectangle in our sums with trapezoids. Since the area of a trapezoid is (length of base)(average of lengths of heights), we end up with the estimate

$T(f, n) = \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\Delta x = \frac{1}{2}(\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x)$
$= \frac{1}{2}(L(f, n) + R(f, n))$ (trapezoid estimate)

If f is close to being linear on each subinterval (i.e., f'' is not too big), this gives a better estimate of the integral than either of L or R alone. In fact, if $|f''(x)| \leq K$ on $[a, b]$, then

$$|\int_a^b f(x)\ dx - T(f, n)| \leq K\frac{(b-a)^3}{12n^2}$$

This, in practice, leads to very good estimates for the integrals of functions we don't know how to find antiderivatives for. Even for functions that we <u>can</u> find antiderivatives for, this gives a practical way to approximate the <u>values</u> of those antiderivatives (think, e.g., of $\arcsin x$), by approximating the corresponding definite integrals.

Improper integrals

Fund Thm of Calc: $\displaystyle\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a)$, where $F'(x) = f(x)$

Problems: $a = -\infty$, $b = \infty$; f blows up at a or b or somewhere in between
integral is "improper"; usual technique doesn't work. Solution to this:

$$\int_a^\infty f(x)\ \mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\ \mathrm{d}x \qquad \int_{-\infty}^b f(x)\ \mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\ \mathrm{d}x$$

(blow up at a) $\displaystyle\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to a^+}\int_r^b f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b f(x)\ \mathrm{d}x$

(similarly for blowup at b (or both!))

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{s\to b^-} \int_a^s f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\ \mathrm{d}x$$

(blows up at c (b/w a and b)) $\displaystyle\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to c^-} \int_a^r f(x)\ \mathrm{d}x + \lim_{s\to c^+} \int_s^b f(x)\ \mathrm{d}x$

The integral converges if (all of the) limit(s) are finite; otherwise, we say that the integral diverges.

Comparison: $0 \le f(x) \le g(x)$ for all x;

if $\displaystyle\int_a^\infty g(x)\ \mathrm{d}x$ *converges*, so does $\displaystyle\int_a^\infty f(x)\ \mathrm{d}x$

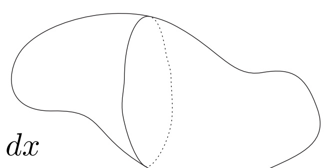
if $\displaystyle\int_a^\infty f(x)\ \mathrm{d}x$ *diverges*, so does $\displaystyle\int_a^\infty g(x)\ \mathrm{d}x$

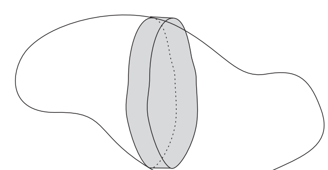
Applications of integration

Volume by slicing. To calculate volume, aprroximate region by objects whose volume we can calculate.

Volume $\approx \sum$(volumes of 'cylinders')
$= \sum$(area of base)(height)
$= \sum$(area of cross-section)Δx_i .

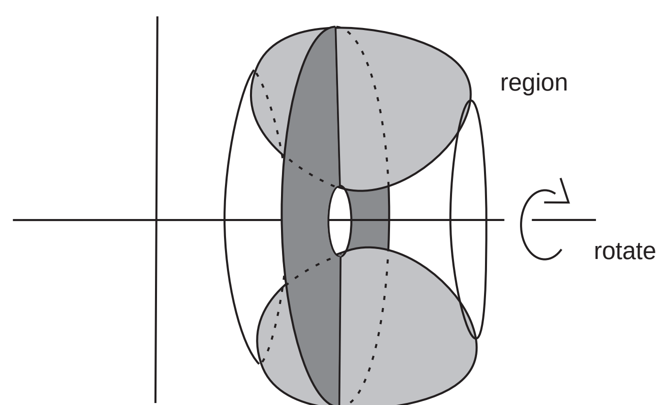
So volume $= \displaystyle\int_{left}^{right}$ (area of cross section) dx





Solids of revolution: disks and washers. Solid of revolution: take a region in the plane and revolve it around an axis in the plane.

take cross-sections perpendicular to axis of revolution;
cross-section = disk (area=πr^2)
or washer (area=$\pi R^2 - \pi r^2$)
rotate around x-axis: write r (and R) as functions of x, integrate dx
rotate around y-axis: write r (and R) as functions of y, integrate dy

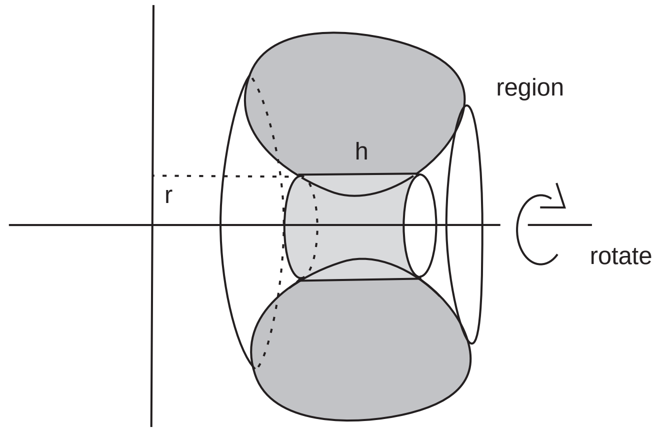


Otherwise, everything is as before: volume $= \displaystyle\int_{left}^{right} A(x)\ dx$ or volume $= \displaystyle\int_{bottom}^{top} A(y)\ dy$

The same is true if axis is parallel to $x-$ or $y-$axis; r and R just change (we add a constant).

Cylindrical shells. Different picture, same volume! Solid of revolution; use cylinders centered on the axis of revolution. The intersection is a cylinder, with area = (circumference)(height) = $2\pi rh$

volume = $\int_{left}^{right}$ (area of cylinder) dx or $\int_{bottom}^{top}$(area of cylinder) dy !



revolve around vertical line:
integrate dx
revolve around horizontal line:
integrate dy

Ex: region in plane between
$y = 4x$, $y = x^2$, revolved around y-axis

left=0, right=4, $r = x$, $h = (4x - x^2)$ volume = $\int_0^4 2\pi x(4x - x^2)\ dx$

Arclength. Idea: approximate a curve by lots of short line segments; length of curve ≈ sum of lengths of line segments.

Line segment between $(c_i, f(c_i))$ and $(c_{i+1}, f(c_{i+1}))$ has length

$$\sqrt{1 + (\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i})^2} \cdot (c_{i+1} - c_i) \approx \sqrt{1 + (f'(c_i))^2} \cdot \Delta x_i$$

<u>So</u> length of curve = $\int_{left}^{right} \sqrt{1 + (f'(x))^2}\ dx$

The problem: integrating $\sqrt{1 + (f'(x))^2}$! Sometimes, $1+(f'(x))^2$ turns out to be a perfect square.....

More generally, we can work with *parametric curves* $(x(t), y(t))$ [think: t = time, so we are travelling around the x-y plane].

Arclength: we approximate it the same way, as a sum of lengths of line seqments that approximate the curve. Each segment has length

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x/\Delta t)^2 + (\Delta y/\Delta t)^2}\Delta t \approx \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

so the length of the curve is $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$