Math 107
Topics for the second exam

(Technically, everything covered on the first exam plus...)

Exponential growth and decay

In many situations, the rate of change of some quantity depends in a known way on the values of the quantity. A basic example is *radioactive decay*: if $f(t)$ is the amount of isotope at time t, then $f'(t) = kf(t)$ for some constant k (which depends upon the isotope). Such equation is called a *differential equation*, since it involves an (unknown) function as well as its derivative.

The equation for radiactive decay is one of a class of equations called *separable* equations. A differential equation is separable if it can be written as $y' = A(t)B(y)$

This allows us to 'separate the variables' and integrate with respect to dy and dt to get a solution:

$$\frac{1}{B(y)}dy = A(t)\ dt \text{ ; integrate both sides}$$

In the end, our solutions look like $F(y) = G(t) + c$, so it defines y *implicitly* as a function of t , rather than explicitly. In some cases we can invert F to get an explicit solution, but often we cannot.

For example, the separable equation $y' = ty^2$, $y(1) = 2$ has solution $\displaystyle\int \frac{dy}{y^2} = \int t\ dt + c$

so solving the integrals we get $(-1/y) = (t^2/2) + c$, or $y = -2/(t^2 + 2c)$; setting $y = 2$ when $t = 1$ gives $c = -1$.

Applying this approach to a radioactive decay problem, $y' = ky$, yields $y(t) = Ce^{kt}$, where the constant of integration C can be determined by setting $t = 0$; $y_0 = y(0) = Ce^0 = C$. So $y(t) = y_0e^kt$. The constant k can then be determined if we know the value of $y(t)$ for any other time t_0; $k = \dfrac{1}{t_0}\ln[y(t_0)/y_0]$.

Newton's Law of Cooling: This states that the rate of change of the temperature $T(t)$ of an object is proportional to the difference between its temperature and the ambient temperature of the air around it. The constant of proportionality depends upon the particular object (and the medium, e.g., air or water) it is in. In other words,

$$T' = k(A - T)$$

Since a cold object will warm up, and a warm object will cool down, this means that the constant k should be positive. This equation is separable, and we can find the solution

$$T(t) = A + (T(0) - A)e^{-kt}$$

Typically, k is not given, but can be determined by knowing the temperature at some other time t_1, by plugging into the equation above and solving for k.

Infinite sequences and series

Limits of sequences of numbers

A sequence is: a string of numbers; a function $f{:}\mathbf{N}\to\mathbf{R}$; write $f(n) = a_n$

$a_n = n$-th term of the sequence

Basic question: convergence/divergence $\lim\limits_{n\to\infty} a_n = L$ (or $a_n \to L$) if

eventually all of the a_n are always as close to L as we like, i.e. for any $\epsilon > 0$, there is an N so that if $n \geq N$ then $|a_n - L| < \epsilon$ Ex.: $a_n = 1/n$ converges to 0 ; can always choose N=$1/\epsilon$

$a_n = (-1)^n$ diverges; terms of the sequence never settle down to a single number

If a_n is increasing ($a_{n+1} \geq a_n$ for every n) and bounded from above ($a_n \leq M$ for every n, for some M) , then a_n converges (but not necessarily to M !)

limit is smallest number bigger than all of the terms of the sequence

Limit theorems for sequences

Idea: limits of sequences are a lot like limits of functions. If $a_n \to L$ and $b_n \to M$, then $(a_n + b_n \to L + M$ $(a_n - b_n) \to L - M$ $(a_nb_n) \to LM$, and

$(a_n/b_n) \to L/M$ (provided M, all b_n are $\neq 0$)

Sqeeze play theorem: if $a_n \leq b_n \leq c_n$ (for all n large enough) and $a_n \to L$ and $c_n \to L$, then $b_n \to L$

If $a_n \to L$ and f:$\mathbf{R} \to \mathbf{R}$ is continuous at L, then $f(a_n) \to f(L)$

if $a_n = f(n)$ for some function f:$\mathbf{R} \to \mathbf{R}$ and $\lim_{x\to\infty} f(x) = L$, then $a_n \to L$

(allows us to use L'Hopital's Rule!)

Another basic list: (x = fixed number, k = konstant)

$\frac{1}{n} \to 0$ $k \to k$ $x^{\frac{1}{n}} \to 1$ $n^{\frac{1}{n}} \to 1$ $(1+\frac{x}{n})^n \to e^x$

$\frac{x^n}{n!} \to 0$ $x^n \to \{$ 0, if $|x| < 1$; 1, if $x = 1$; diverges, otherwise $\}$

Infinite series

An infinite series is an infinite sum of numbers

$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$ (summation notation)

n-th term of series $= a_n$; N-th partial sum of series $= s_N = \sum_{n=1}^{N} a_n$

An infinite series **converges** if the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges

We may start the series anywhere: $\sum_{n=0}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=3437}^{\infty} a_n$, etc. ;

convergence is unaffected (but the number it adds up to is!)

Ex. geometric series: $a_n = ar^n$; $\sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$ **if** $|r| < 1$; otherwise, the series **diverges**.

Ex. Telescoping series: partial sums s_N 'collapse' to a simple expression

E.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})$; $s_N = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - (\frac{1}{N+1} + \frac{1}{N+2}))$

n-th term test: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

So if the n-th terms **don't** go to 0, then $\sum_{n=1}^{\infty} a_n$ diverges

Basic limit theorems: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty}(ka_n) = k\sum_{n=1}^{\infty} a_n$ Truncating a series: $\sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \sum_{n=1}^{N-1} a_n$

Comparison tests

Again, think $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ all n

Convergence depends only on partial sums s_N being **bounded**

One way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all n (past a certain point), then

if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$; if $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: a_n and $b_n \geq 0$ for all n, $\frac{a_n}{b_n} \to L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ anf $\sum b_n$ either **both** converge or **both** diverge

If $L=0$ and $\sum b_n$ converges, then so does $\sum a_n$; If $L=\infty$ and $\sum b_n$ diverges, then so does $\sum a_n$.
(Why? eventually $(L/2)b_n \le a_n \le (3L/2)b_n$; so can use comparison test.)

Ex: $\sum 1/(n^3-1)$ converges; L-comp with $\sum 1/n^3$; $\sum n/3^n$ converges; L-comp with $\sum 1/2^n$
$\sum 1/[n\ln(n^2+1)]$ diverges; L-comp with $\sum 1/(n\ln n)$

The integral test

Idea: $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n \ge 0$ all n, then the partial sums
$\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence; so converges exactly when bounded from above

If (eventually) $a_n = f(n)$ for a **decreasing** function $f:[a,\infty)\to\mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x)\ \mathrm{d}x \le s_N = \sum_{n=a}^{N} a_n \le \int_a^N f(x)\ \mathrm{d}x$$

so $\displaystyle\sum_{n=a}^{\infty} a_n$ converges exactly when $\displaystyle\int_a^{\infty} f(x)\ \mathrm{d}x$ converges

Ex: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges exactly when $p>1$ (p-series)

Absolute convergence and alternating series

A series $\sum a_n$ <u>converges</u> <u>absolutely</u> if $\sum |a_n|$ converges. If $\sum |a_n|$ converges then $\sum a_n$ converges. A series which converges but does not converge absolutely is called *conditionally convergent.*

An *alternating series* has the form $\sum(-1)^n a_n$ with $a_n \ge 0$ for all n.

If the sequence a_n is <u>decreasing</u> and has <u>limit</u> 0, then the **alternating series test** states that $\sum(-1)^n a_n$ converges. For example, $\sum_{n=0}^{\infty}(-1)^n/(n+1)$ converges, but not absolutely, so it is conditionally convergent.

The ratio and root tests

Previous tests have you compare your series with **something else** (another series, an improper integral); these tests compare a series with itself (sort of)

Ratio Test: $\sum a_n$, $a_n \ne 0$ all n; $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

If $L<1$ then $\sum a_n$ converges absolutely If $L>1$, then $\sum a_n$ diverges
If $L=1$, then try something else!

Root Test: $\sum a_n$, $\displaystyle\lim_{n\to\infty}|a_n|^{1/n} = L$

If $L<1$ then $\sum a_n$ converges absolutely If $L>1$, then $\sum a_n$ diverges
If $L=1$, then try something else!

Ex: $\displaystyle\sum\frac{4^n}{n!}$ converges by the ratio test $\displaystyle\sum\frac{n^5}{n^n}$ converges by the root test

Power series

Idea: turn a series into a function, by making the terms a_n <u>depend</u> on x
replace a_n with $a_n x^n$; series of powers

$\displaystyle\sum_{n=0}^{\infty} a_n x^n$ = power series centered at 0 $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ = power series centered at a

Big question: for what x does it converge? Solution from ratio test: $\displaystyle\lim\left|\frac{a_{n+1}}{a_n}\right| = L$, set $R=\dfrac{1}{L}$

then $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for $|x-a|<R$ and diverges for $|x-a|>R$;

R = radius of convergence Ex.: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; conv. for $|x|<1$

Why care about power series?

Idea: partial sums $\sum_{k=0}^{n} a_k x^k$ are <u>polynomials</u>;

if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then the poly's make good approximations for f

Differentiation and integration of power series

Idea: if you differentiate or integrate each term of a power series, you get a power series which is the derivative or integral of the original one.

If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of conv R,

then so does $g(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$, and $g(x) = f'(x)$

<u>and</u> so does $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$, and $g'(x) = f(x)$

Ex: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $f'(x) = f(x)$, so (since $f(0) = 1$) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ex.: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ (for $|x| < 1$), so (replacing x with $-x$)

$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$, so (replacing x with $x-1$) $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$

Ex:. $\arctan x = \int \frac{1}{1-(-x^2)}\,\mathrm{d}x = \int \sum_{n=0}^{\infty} (-x^2)^n\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ (for $|x| < 1$)

Taylor series

Idea: start with function $f(x)$, find power series <u>for</u> it.

If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, then (term by term diff.) $f^{(n)}(a) = n! a_n$; So $a_n = \frac{f^{(n)}(a)}{n!}$

Starting with f, define $P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, the Taylor series for f, centered at a.

$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$, the n-th Taylor polynomial for f.

Ex.: $f(x) = \sin x$, then $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

Big questions: Is $f(x) = P(x)$? (I.e., does $f(x) - P_n(x)$ tend to 0 ?)
If so, how well do the P_n's approximate f ? (I.e., how small is $f(x) - P_n(x)$?)

Error estimates

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

means that the value of f at a point x (far from a) can be determined just from the behavior of f near a (i.e., from the derivs. of f at a). This is a very powerful property, one that we wouldn't ordinarily expect to be true. The amazing thing is that it often is:

$$P(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \; ; \; P_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \; ;$$

$R_n(x,a)= f(x) - P_n(x,a) = n$-th remainder term = error in using P_n to approximate f

Taylor's remainder theorem : estimates the size of $R_n(x,a)$

If $f(x)$ and all of its derivatives (up to $n+1$) are continuous on $[a,b]$, then

$f(b) = P_n(b,a) + \dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$, for some c in $[a,b]$

i.e., for each x, $R_n(x,a) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some c between a and x

so if $|F^{(n+1)}(x)| \leq M$ for every x in $[a,b]$, then $|R_n(x,a)| \leq \dfrac{M}{(n+1)!}(x-a)^{n+1}$ for every x in $[a,b]$

Ex.: $f(x)$=$\sin x$, then $|f^{(n+1)}(x)| \leq 1$ for all x, so $|R_n(x,0)| \leq \dfrac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

so $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ Similarly, $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

Use Taylor's remainder to estimate values of functions:

$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}$, so e=e^1=$\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n)!}$

$|R_n(1,0)| = \dfrac{f^{(n+1)}(c)}{(n+1)!} = \dfrac{e^c}{(n+1)!} \leq \dfrac{e^1}{(n+1)!} \leq \dfrac{4}{(n+1)!}$

since $e < 4$ (since $\ln(4) > (1/2)(1) + (1/4)(2) = 1$) (Riemann sum for integral of $1/x$)

so since $\dfrac{4}{(13+1)!} = 4.58{\times}10^{-11}$,

$e = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \cdots + \dfrac{1}{13!}$, to 10 decimal places.

Other uses: if you know the Taylor series, it tells you the values of the derivatives at the center.

Ex.: e^x=$\displaystyle\sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}$, so $xe^x = \displaystyle\sum_{n=0}^{\infty} \frac{(x)^{n+1}}{(n)!}$, so

the 15th deriv of xe^x , at 0, is 15!(coeff of x^{15}) $= \dfrac{15!}{14!} = 15$

Substitutions: new Taylor series out of old ones

Ex. $\sin^2 x = \dfrac{1-\cos(2x)}{2} = \dfrac{1}{2}(1 - \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n}}{(2n)!}$

$= \dfrac{1}{2}(1 - (1 - \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} - \dfrac{(2x)^6}{6!} + \cdots \quad = \dfrac{2x^2}{2!} - \dfrac{2^3x^4}{4!} + \dfrac{2^5x^6}{6!} - \dfrac{2^7x^8}{8!} + \cdots$

Integrate functions we can't handle any other way:

Ex.: $e^{x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(x)^2n}{(n)!}$, so $\displaystyle\int e^{x^2}\ \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(x)^{2n+1}}{n!(2n+1)}$