Math 107

Topics since the third exam's review sheet

Note: The final exam covers topics from all of our review sheets for the semester.

Work

In physics, one studies the behavior of objects when acted upon by various *forces*. Newton's Laws provide the basic connection between a force acting on an object and the effect it has on its motion:

$$F = ma \; ; \qquad \text{Force} = \text{mass} \times \text{acceleration}$$

Two basic quantites to compute, when you know the force, are *impulse* and *work*.

Impulse measures the effect of a force over time. If a constant force F is applied to an object, over a time interval of length T, then the impulse imparted to the object is Impulse $= J = F \cdot T$. But typically the force being apllied will not be constant. Then we do what we usually do: look at the impulse generated by the force over a short time interval (where ther force is effectively constant), and add up the impulses imparted over all of these little intervals.

$J \approx \sum F(t_i)\,\Delta t$, which looks suspiciously like an integral. So we define $J = \int_0^T F(t)\,dt$

But in classical physics, where $F(t) = m \cdot a(t) = m \cdot x''(t)$, if we can treat m as a constant, then we can integrate F, so

$$J = m \cdot x'(T) - m \cdot x'(0) = m \cdot v(T) - m \cdot v(0)$$

is the change of momentum of the object.

In physics, *work* represents force being applied across a distance. If a constant force F is applied to an object, which moves the object a distance D, then the work done on the object is $W = F \cdot D$. Again, if the force applied across this distance is not constant, then we interpret work, in stead, as an integral, by cutting the distance covered into small pieces of length δx :

$$W \approx \sum F(x_i)\ \Delta x \text{ , so } W = \int_0^D F(x)\ dx$$

An interesting application of these ideas comes when trying to compute the amount of work necessary to pump out a tank of some known shape. If the tank has height D (we will think of the top of the tank as being at $x = D$ and the bottom being at $x = 0$), and at height x our cross-section of the tank has area $A(x)$, then if (as when we computed volume) we think of the fluid in the tank as being a stack of cylinders with height Δx, the work necessary to lift the slice at height x to the top of the tank will be

$$W = (\text{force})(\text{distance}) = (m \cdot g) \cdot (D - x) = ((A(x) \cdot \Delta x)\rho g) \cdot (D - x)$$

where ρ is the density of the fluid, m = mass = (volume)(density), and g is the acceleration due to gravity (which provides the force we need to overcome to push the fluid up out of the tank), and $D - x$ is the distance that the slice must be lifted. Therefore, the work done to empty the tank is approximated by a sum of such quantities, which in turn models a definite integral; the work done in emptying the tank is

$$W = \rho g \int_0^D (D - x)A(x)\ dx$$

In English units, ρg is typically reported in pounds per cubic foot, $A(x)\ dx$ has units of cubic feet, and $D - x$ is feet, so the integral has units of foot-pounds.

Math 107: a checklist of topics encountered

- Techniques of integration
  - u-Substitution
  - Integration by Parts
  - Trigonometric Substitution
  - Trigonometric Integrals
    - reduction formulas
  - Partial Fractions
  - Numerical Integration
    - trapezoid rule, error estimates
  - Improper Integrals
    - convergence, (limit) comparison theorem
- Applications of Integration
  - Volume by Slicing
  - Volume by Cylindrical Shells
  - Arclength
  - Exponential Growth and Decay
  - Work
- Sequences and Series
  - n-th Term Test
  - Geometric Series
    - exact calculation of sums
  - Comparison/Limit Comparison Tests
  - Integral Test
  - Ratio(n) and Root Tests
  - Absolute and Conditional Convergence
    - alternating series test, remainder estimate
  - Power Series
    - radius of convergence, term-by-term differentiation and integration
  - Taylor Series
    - Taylor's Theorem, remainder estimates
- Polar Coordinates
  - Polar coordinates vs. Cartesian coordinates
  - Polar Curves
    - slopes of tangents
    - area enclosed by curve
    - arclength
- Vectors
  - Coordinate representation, Length
  - Angle, Dot product
    - orthogonal projection
  - Lines in 3-dimensional space
    - distance from a point to a line
- Vector-valued Functions
  - Parametrized curves in space
  - Differeniation and Antidifferentiation
    - position, from velocity, from acceleration
  - Arclength